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UNIVERSITY OF GLOBAL VILLAGE (UGV), BARISHAL.

THE FIRST SKILL BASED HI-TECH UNIVERSITY IN BANGLADESH

Fourier Analysis and Laplace Transform

Lecture Prepared By

Md. Shakibul Ajam

Lecturer

Department of Mathematics

University of Global Village (UGV), Barishal

Course Name: Fourier Analysis and Laplace Transform Course Code: MAT 0541-2105 Semester End Exam (SEE): 3 Hours	Credit: 03 CIE Marks: 90 SEE marks: 60

Course Learning Outcomes (CLO): After successful completion of the course students will be able to -

CLO1	Define the basic terminology and theorems associated with Fourier Analysis and Laplace Transformation.
CLO2	Properties of Laplace and Inverse Laplace Transformation and Laplace Transformation of derivatives, and Applications.
CLO3	Describe Fourier Series, Fourier Sine and Cosine Series, Orthogonal Functions, Fourier Integrals.
CLO4	Apply the acquired concepts of Fourier Analysis, and Laplace Transformation in engineering.

Course Content Summary

SL.	Content of Courses	Hrs	CLO's
1	Definition of Laplace transformations, Some important properties of Laplace Transformations, and some related mathematics, Laplace transformations of some elementary functions, Laplace Transformation of 1st and 2nd derivatives and general term derivatives, multiplication by t power n and division by t, Inverse Laplace transformations	8	CLO1, CLO2
2	Inverse Laplace transformations of some elementary functions, Inverse Laplace Transformation of 1st and 2nd derivatives and, Ordinary Differential Equations with Constant Coefficients, Related Mathematics. Applications to electrical circuits, L-R circuit related Problems.	6	CLO1, CLO2
3	Definition of Fourier series, Periodic Function, Even and Odd Function, Piecewise Continuous Functions, Dirichlet Conditions, Parsivals Identity, Fourier Series, Some important properties of Fourier series, Half range Fourier Sine or Cosine Series, and Related mathematics, Convergence of Fourier series, Definition of orthogonal Functions,Orthogonality, Orthogonal series	10	CLO3, CLO1
4	Application of Fourier series in engineering, Boundary value Problem, Laplace equation and Related mathematics,Fourier integrals, Fourier Transforms, Fourier sine and cosine Transforms, Convolution Theorem, Application of Fourier integrals, Related mathematics.	10	CLO4, CLO1

Course Plan Specifying content, CLO's, Teaching Learning, and Assessment strategy mapping with CLO's

Week	Topics	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLO's
1	Laplace transformation • Definition • Notation • Proof of formulas of Laplace Transformations	Lecture, Discussion	Quiz	CLO1, CLO2
2	Some important properties of Laplace Transformations • Linearity property • Change of scale property • Related mathematics	Discussion, Oral Presentation	Written Assignment	CLO1, CLO2
3	Multiplication by t power n Derivatives of Laplace Transformation • 1st and 2nd derivatives • General term derivatives • Related mathematics	Oral Presentation	Oral Presentation	CLO1, CLO2
4	Inverse Laplace transformation • Definition • Notation Proof of formulas of Laplace Transformations	Group Work	Group Assignment	CLO1, CLO2
5	Some important properties of Inverse Laplace Transformations • Linearity property • Change of scale property • Related mathematics	Case Study	Presentation	CLO1, CLO2

Course Plan Specifying content, CLO's, Teaching Learning, and Assessment strategy mapping with CLO's

Week	Topics	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLO's
6	Ordinary Differential Equations with Constant Coefficients • Initial and boundary Value problem • Related mathematics	Group Work	Quiz, Written Assignment	CLO1, CLO2
7	Applications to electrical circuits • L-R circuit • Related problems	Lecture, Discussion	Oral Presentation, Quiz	CLO4, CLO2
8	Applications to beam. • formulation • Related problems	Discussion, Oral Presentation	Group Assignment, Quiz	CLO3, CLO1
9	Fourier series • Definition • Periodic Function • Piecewise function • Odd and even functions	Oral Presentation	Presentation, Written Assignment	CLO3, CLO1
10	Graph of different types of functions • Even and odd functions • Periodic functions	Oral Presentation	Quiz, Presentation	CLO3, CLO5
11	Some important properties of Fourier Series • Dirichlet Conditions • Parseval's Identity • Theorems and related mathematics	Group Work	Written Assignment,	CLO3, CLO5

Course Plan Specifying content, CLO's, Teaching Learning, and Assessment strategy mapping with CLO's

Week	Topics	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLO's
12	Fourier series of different types of functions Related mathematics.	Discussion, Oral Presentation	Group Assignment, Presentation	CLO4, CLO1
13	Half range Fourier Sine or Cosine Series • Definition • Related mathematics.	Discussion, Oral Presentation	Quiz, Group Assignment	CLO4, CLO1
14	• Fourier integrals • Convolution Theorem • mathematics.	Oral Presentation	Written Assignment, Quiz	CLO4, CLO1
15	Orthogonality • Definition of orthogonal Functions • Orthogonality, • Related mathematics	Lecture, Discussion	Oral Presentation, Group Assignment	CLO4, CLO1
16	Application • Boundary value Problems • Laplace equation • Related mathematics.	Practical Work	Presentation, Quiz	CLO4, CLO1
17	Application of Fourier Series in Engineering • mathematical problems	Reading Assignment	Quiz, Written Assignment, Oral Presentation	CLO4, CLO1

Assessment pattern

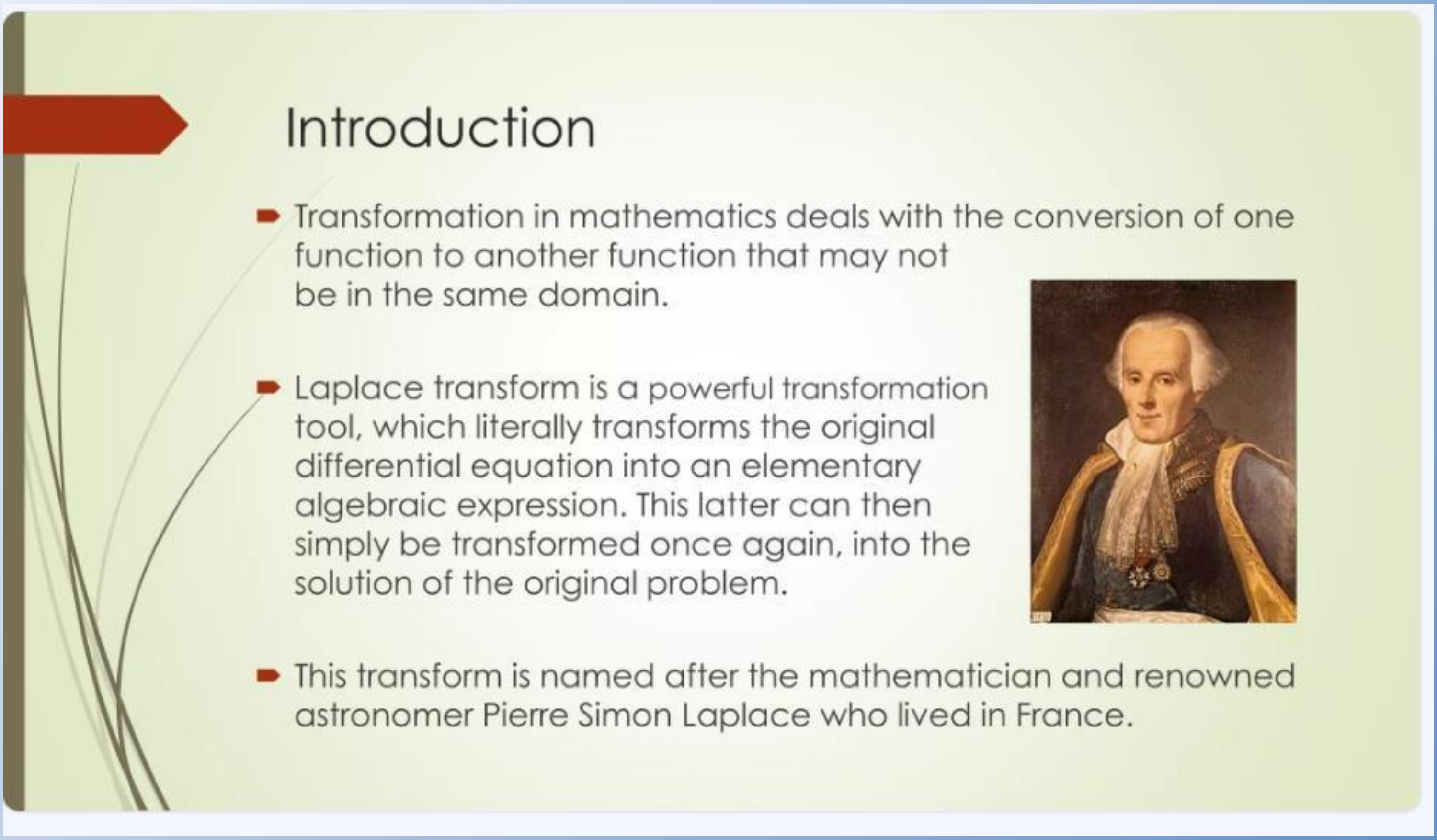
CIE- Continuous Internal Evaluation (90 Marks)	
Bloom's Category	Test
Remember	10
Understand	10
Apply	10
Analyze	10
Evaluate	15
Create	5

SEE- Semester End Examination (60 Marks)				
Bloom's Category Marks (out of 60)	Tests (45)	Assignments (15)	Quizzes (15)	Attendance (15)
Remember	05			
Understand	05		05	
Apply	10	05	05	15
Analyze	10	05	05	
Evaluate	10	05		
Create	05			

Week 1

Topics: Laplace Transformation

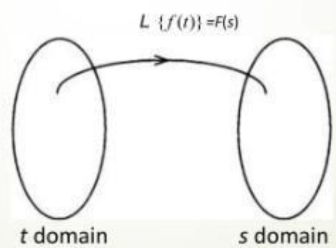
Pages (7-11)



Definition of Laplace Transform

Suppose that, f is a real or complex valued function of the variable $t > 0$ and s is a real or complex parameter. We define the Laplace transform of f as

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$



Laplace Transformation: Let *F(t)* be a function of t specified for *t>0*. Then the Laplace Transform of *F(t)*, denoted by $\mathcal{L}\{F(t)\}$ is define by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$,where the parameter s is real.

Some formula of Laplace Transformation

$$(i)\ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad (ii)\ \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad s > a$$

$$(iii)\ \mathcal{L}\{cos\ at\} = \frac{s}{s^2 + a^2}, \quad s > 0 \qquad (iv)\ \mathcal{L}\{sin\ at\} = \frac{a}{s^2 + a^2}, \qquad s > 0$$

Question: Prove that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a},\ s > a$

Solution:

Let, $F(t) = e^{at}$

By the definition of Laplace transformation, we know that

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

So, $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\ dt$

$$= \int_0^\infty e^{-st+at}\ dt \quad = \int_0^\infty e^{-(s-a)t}\ dt$$

$$= \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty \quad = \frac{1}{-(s-a)}(e^{-\infty} - e^0) \quad = \frac{1}{s-a},\ \ s > a$$

Question: Prove that $\mathcal{L}\{1\} = \frac{1}{s},\ s > 0$

Solution:

Let, $F(t) = 1$

By the definition of Laplace transformation, we know that

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\ dt$$

So, $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}.1\ dt$

$$= \int_0^\infty e^{-st}\ dt \quad = \left. \frac{e^{-st}}{-s} \right|_0^\infty \quad = \frac{1}{-s}(e^{-\infty} - e^0) \quad = \frac{1}{s},\ \ s > 0$$

Question: Prove that $\mathcal{L}\{t\} = \frac{1}{s^2},\ s > 0$

Solution:

Let, $F(t) = t$

By the definition of Laplace transformation, we know that

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\ dt$$

So, $\mathcal{L}\{t\} = \int_0^\infty e^{-st}.t\ dt$

$$=\int_0^\infty e^{-st} t\, dt \quad = t\int_0^\infty e^{-st}\, dt - \int_0^\infty \left\{ \frac{dt}{dt} \int_0^\infty e^{-st}\, dt \right\} dt$$

$$= t\left. \frac{e^{-st}}{-s} \right|_0^\infty - \int_0^\infty \left\{ 1.\frac{e^{-st}}{-s} \right\} dt = 0 - \left(\frac{1}{-s} \right). \left. \frac{e^{-st}}{-s} \right|_0^\infty$$

$$= \frac{1}{-s^2}(e^{-\infty} - e^0) \quad = \frac{1}{s^2}, \quad s > 0$$

Question: Prove that $\mathcal{L}\{4\, e^{at}\} = \frac{4}{s-a}, \ s > a$

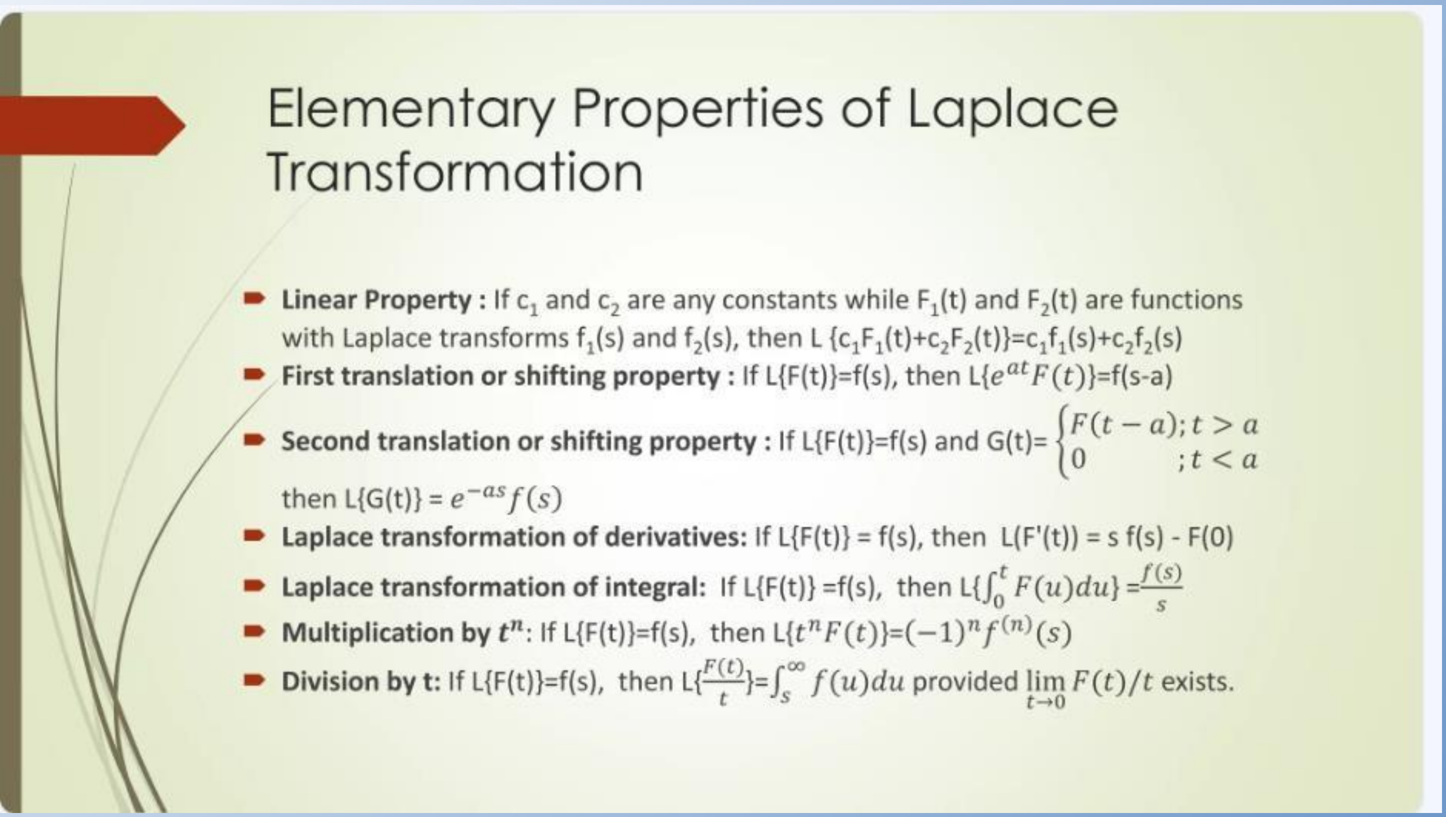
Question: Prove that $\mathcal{L}\{5\} = \frac{5}{s}, \ s > 0$

Question: Prove that $\mathcal{L}\{3t\} = \frac{3}{s^2}, \ s > 0$

Week 2

Topics: Properties of Laplace

Transformation Pages (12-15)



Linear property of Laplace theorem: If c_1 and c_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transformations $f_1(s)$ and $f_2(s)$ respectively, then

$$\mathcal{L}\{c_1F_1(t)+c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F(t)\} = c_1f_1(s) + c_2f_2(s)$$

is called the linear property of Laplace Transformations. The result is easily extended for n terms as follows:

$$\mathcal{L}\{c_1F_1(t)+c_2F_2(t)+\cdots+c_nF_n(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\} + \cdots + c_n\mathcal{L}\{F_n(t)\}$$

Or,

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} + \cdots + c_n \mathcal{L}\{f_n(t)\}$$

<u>Questions-1:</u> Find the Laplace transformation of $4e^{5t} + 6t^3 - 3\sin 4t + 5\cos 2t$

<u>Solution:</u> By linear property of Laplace transformations, we have

$$\mathcal{L}\{4e^{5t} + 6t^3 - 3\sin 4t + 5\cos 2t\}$$

$$= 4\mathcal{L}\{e^{5t}\} + 6\mathcal{L}\{t^3\} - 3\mathcal{L}\{\sin 4t\} + 5\mathcal{L}\{\cos 2t\}$$

$$= 4\left(\frac{1}{s-5}\right) + 6\left(\frac{3!}{s^4}\right) - 3\left(\frac{4}{s^2+4^2}\right) + 5\left(\frac{s}{s^2+2^2}\right)$$

$$= \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{5s}{s^2+4}$$

<u>Question-2:</u> Find the Laplace transformation of $3t^4 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$.

<u>Solution:</u> By linear property of Laplace transformations, we have

$$\mathcal{L}\{3t^4 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$$

$$= 3\mathcal{L}\{t^4\} + 4\mathcal{L}\{e^{-3t}\} - 2\mathcal{L}\{\sin 5t\} + 3\mathcal{L}\{\cos 2t\}$$

$$= 3\left(\frac{4!}{s^5}\right) + 4\left(\frac{1}{s+3}\right) - 2\left(\frac{5}{s^2+5^2}\right) + 3\left(\frac{s}{s^2+2^2}\right)$$

$$= \frac{72}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3\,s}{s^2+4}$$

<u>Question:</u> Find the Laplace transformation of the followings:

(i) $7t^4 + 5e^{-6t} - 4\sin 5t + 2\cos 2t$

(ii) $3t^3 + 4e^{-5t} + 3\cos 4t - 2\sin 6t$

(iii) $10t^6 - 15e^{10t} + 12\sin t + 6\cos 6t$

(iv) $10\sin 10t - 12t^7 - 2\cos t + e^{-t}$

(v) $10t^3 - 5e^{-7t} - 20\sin 6t + 20\cos 7t$

(vii) $13e^{10t} + 6e^{-t} + 12t^8 + 6\sin t + 2\cos 9t$

(viii) $2e^{-5t} + 7e^{6t} + t^7 - 2\sin 8t + 7\cos t$

Change of scale or shifting property:

If $\{F(t)\} = f(s)$, then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$

Questions: Find the Laplace transformation of the expression$e^{-2t}(3\cos 6t - 5\sin 6t)$.

Solution: We have $\mathcal{L}\{3\cos 6t - 5\sin 6t\}$

$$= 3\left(\frac{s}{s^2+6^2}\right) - 5\left(\frac{6}{s^2+6^2}\right)$$

$$= \frac{3s}{s^2+36} - \frac{30}{s^2+36} = \frac{3s-30}{s^2+36}$$

Then $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = \frac{3(s+2)-30}{(s+2)^2+36} = \frac{3s-24}{s^2+4s+40}$

Question-2: Find the Laplace transformation of

$(i) e^{2t}(3\cos 4t - 4\sin 4t)$ $(ii) e^{-4t}(6\sin 3t - 5\cos 3t)$

Solution: (i) We have $\mathcal{L}\{3\cos 4t - 4\sin 4t\}$

$$= 3\left(\frac{s}{s^2+4^2}\right) - 5\left(\frac{4}{s^2+4^2}\right)$$

$$= \frac{3s}{s^2+16} - \frac{20}{s^2+16} = \frac{3s-20}{s^2+16}$$

Then $\mathcal{L}\{e^{2t}(3\sin 4t - 4\cos 4t)\} = \frac{3(s-2)-30}{(s-2)^2+36} = \frac{3s-6-30}{s^2-4s+4+36} = \frac{3s-6-3}{s^2-4s+40}$

(ii) We have $\mathcal{L}\{6\, sin3t - 5\cos 3t\}$

$$= 6\left(\frac{3}{s^2+3^2}\right) - 5\left(\frac{s}{s^2+3^2}\right)$$

$$= \frac{18}{s^2+9} - \frac{5s}{s^2+9} = \frac{18-5s}{s^2+9}$$

Then $\mathcal{L}\{e^{-4t}(6\, sin3t - 5\cos 3t)\} = \frac{18-5(s+4)}{(s+4)^2+9} = \frac{18-5s-20}{s^2+16s+16+9} = \frac{-(5s+2)}{s^2+16s+25}$

Question: Find the Laplace transformation of the followings:

(i) $2e^{-3t}(\sin 4t)$, $e^{-t}\cos 9t$

(ii) $e^{-3t}(6t^3 - 7\cos t + 4\sin t)$

(iii) $e^{-4t}(6\sin 3t - 5\cos 3t)$

(iv) $e^{-t}(\cos 2t + 2t^2)$

(v) $e^{4t}(3t^4 + 2\sin 7t + \cos 5t)$

(vi) $e^{-5t}(4t^3 + 3\cos t - 4\sin 6t)$

(vii) $e^{3t}(7\cos 3t + 4\sin 6t)$

Week 3
Topics: Laplace Transformation of derivatives
Pages (16-18)

Laplace Transformation of derivatives:

If $\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$

and $\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

Multiplication by powers of t:

If $\{F(t)\} = f(s)$, then $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

Question: Find $\mathcal{L}*t\ sinat\}$.

Solution: Since we know that,

$$\mathcal{L}\{sinat\} = \frac{a}{s^2 + a^2}$$

So,

$$\mathcal{L}\{t\ sinat\} = \{(-1)^1\} \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{2as}{(s^2 + a^2)^2}$$

Question: Find $\mathcal{L}\{t^2 e^{2t}\}\ and\ \mathcal{L}*t\ cosat\}$.

LAPLACE TRANSFORM OF DERIVATIVES

13. Prove *Theorem 1-6*: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = s f(s) - F(0)$.

Using integration by parts, we have

$$\begin{aligned}
\mathcal{L}\{F'(t)\} &= \int_0^\infty e^{-st} F'(t)\,dt = \lim_{P\to\infty} \int_0^P e^{-st} F'(t)\,dt \\
&= \lim_{P\to\infty} \left\{ e^{-st} F(t) \Big|_0^P + s\int_0^P e^{-st} F(t)\,dt \right\} \\
&= \lim_{P\to\infty} \left\{ e^{-sP} F(P) - F(0) + s\int_0^P e^{-st} F(t)\,dt \right\} \\
&= s\int_0^\infty e^{-st} F(t)\,dt - F(0) \\
&= s f(s) - F(0)
\end{aligned}$$

using the fact that $F(t)$ is of exponential order γ as $t \to \infty$, so that $\lim_{P\to\infty} e^{-sP} F(P) = 0$ for $s > \gamma$.

For cases where $F(t)$ is not continuous at $t = 0$, see Problem 68.

14. Prove *Theorem 1-9*, Page 4: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F''(t)\} = s^2 f(s) - s\,F(0) - F'(0)$.

By Problem 13,

$$\mathcal{L}\{G'(t)\} = s\,\mathcal{L}\{G(t)\} - G(0) = s\,g(s) - G(0)$$

Let $G(t) = F'(t)$. Then

$$\begin{aligned}
\mathcal{L}\{F''(t)\} &= s\,\mathcal{L}\{F'(t)\} - F'(0) \\
&= s\,[s\,\mathcal{L}\{F(t)\} - F(0)] - F'(0) \\
&= s^2\,\mathcal{L}\{F(t)\} - s\,F(0) - F'(0) \\
&= s^2 f(s) - s\,F(0) - F'(0)
\end{aligned}$$

Problem:

Find (*a*) $\mathcal{L}\{t \sin at\}$, (*b*) $\mathcal{L}\{t^2 \cos at\}$.

(*a*) Since $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2 + a^2}$, we have by Problem 19

$$\mathcal{L}\{t \sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right) = \frac{2as}{(s^2 + a^2)^2}$$

Another method.

Since $$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\, dt = \frac{s}{s^2+a^2}$$

we have by differentiating with respect to the parameter a [using Leibnitz's rule],

$$\frac{d}{da}\int_0^\infty e^{-st}\cos at\, dt = \int_0^\infty e^{-st}\{-t\sin at\}\, dt = -\mathcal{L}\{t\sin at\}$$

$$= \frac{d}{da}\left(\frac{s}{s^2+a^2}\right) = -\frac{2as}{(s^2+a^2)^2}$$

from which

$$\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$

Note that the result is equivalent to $\frac{d}{da}\mathcal{L}\{\cos at\} = \mathcal{L}\left\{\frac{d}{da}\cos at\right\}$.

(*b*) Since $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, we have by Problem 19

$$\mathcal{L}\{t^2\cos at\} = \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right) = \frac{2s^3-6a^2s}{(s^2+a^2)^3}$$

We can also use the second method of part (*a*) by writing

$$\mathcal{L}\{t^2\cos at\} = \mathcal{L}\left\{-\frac{d^2}{da^2}(\cos at)\right\} = -\frac{d^2}{da^2}\mathcal{L}\{\cos at\}$$

which gives the same result.

Week 4

Topics: Inverse Laplace Transformation

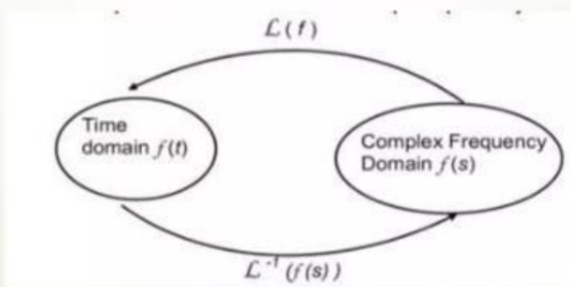
Pages (19-30)

Inverse Laplace Transformation: If the Laplace transformation of a function *F(t)* is *f(s)*, i.e $\mathcal{L}\{F(t)\} = f(s)$, then *F(t)* is called the inverse Laplace transformation of *f(s)*, and we can write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$, where $\mathcal{L}^{-1}$ is called the inverse Laplace operator.

Definition of Inverse Laplace Transform

In order to apply the Laplace transform to physical problems, it is necessary to invoke the inverse transform. If L{f(t)}= F(s), then the inverse Laplace Transform is denoted by

$$L^{-1}\{F(s)\} = f(t), \quad t \geq 0$$



Some formula of inverse Laplace Transformation

$$(i)\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!} \qquad (ii)\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$(iii)\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = cos\ at \qquad (iv)\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = sin\ at$$

Linear property of inverse Laplace transformation: If c_1 and c_2 are any constants while $f_1(s)$ and $f_2(s)$ are functions with Laplace transformations $F_1(t)$ and $F_2(t)$ respectively, then

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\}$$

is called the linear property of inverse Laplace transformation. And for n times we can write,

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s) + \cdots + c_n f_n(s)\} = c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\} + \cdots + c_n\mathcal{L}^{-1}\{f_n(s)\}$$

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s) + \cdots + c_n f_n(s)\} = c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\} + \cdots + c_n\mathcal{L}^{-1}\{f_n(s)\}$$

Questions-1: Find the inverse Laplace transformation of the expression $\frac{1}{s^5} + \frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}$

Solution: By linear property of Laplace transformations, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5} + \frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{3s}{s^2+16}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{s^2+4}\right\}$$

$$= \frac{1}{24}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \frac{1}{24} t^4 + 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

Question-2: Find the inverse Laplace transformation $\frac{5s+4}{s^5} + \frac{6}{s-3} - \frac{8s}{s^2+9} + \frac{7}{s^2+25}$

Solution: By linear property of Laplace transformations, we have

$$\mathcal{L}^{-1}\left\{\frac{5s+4}{s^5} + \frac{6}{s-3} - \frac{8s}{s^2+9} + \frac{7}{s^2+25}\right\}$$

$$= 5\,\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} + 4\,\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{8s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{7}{s^2+25}\right\}$$

$$= \frac{5}{6}\mathcal{L}_{-1}\left\{\frac{3!}{s^{3+1}}\right\} + \frac{4}{24}\mathcal{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\} + 6\left\{\frac{1}{s-3}\right\} - 8\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\}$$

$$= \frac{5}{6}t^3 + \frac{1}{6}t^4 + 6\,e^{3t} - 8\cos 3t + \frac{7}{5}\sin 5t$$

Question: Find the Inverse Laplace transformation of the followings:

(i) $\frac{6}{2s-3} - \frac{3+4s}{9s^2+16} + \frac{8-6s}{16s^2+9}$

(ii) $\frac{1}{s^5} + \frac{4}{s-2} + \frac{3s}{s^2+16} + \frac{5}{s^2+4}$

(iii) $\frac{5s+4}{s^5} + \frac{6}{s-3} - \frac{8s}{s^2+9} + \frac{7}{s^2+25}$

(iv) $\frac{3}{s+4} - \frac{2s+5}{s^2+16}$

(v) $\frac{s+10}{s^4} - \frac{4}{s-6} + \frac{s+8}{s^2+4}$

Change of scale property:

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then $\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$

Questions: Find the inverse Laplace transformation of the expression $\frac{6s-4}{s^2-4s+20}$.

Solution: We have $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = \mathcal{L}^{-1}\left\{\frac{6s-12+8}{s^2-2.s.2+2^2+16}\right\} = \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$

$$= 6\,\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+16}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+16}\right\}$$

$$= 6\,e^{2t}\cos 4t + 2\,e^{2t}\sin 4t$$

Question-2: Find the inverse Laplace transformation of $(i)\ \frac{4s-25}{s^2-6s+34}$ $\quad (ii)\ \frac{6s-4}{s^2-2s+10}$

Solution: We have $\mathcal{L}^{-1}\left\{\frac{4s-25}{s^2-6s+34}\right\} = \mathcal{L}^{-1}\left\{\frac{4s-12-13}{s^2-2.s.3+3^2+25}\right\} = \mathcal{L}^{-1}\left\{\frac{4(s-3)-13}{(s-3)^2+5^2}\right\}$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{(s-3)}{(s-3)^2+5^2}\right\} - \frac{13}{5}\mathcal{L}^{-1}\left\{\frac{5}{(s-3)^2+5^2}\right\}$$

$$= 4\ e^{3t}\cos 5t - \frac{13}{5}\ e^{3t}\sin 5t$$

Result on inverse Laplace transform

Result: 1 Linear property

$L[f(t)] = F(s)$ and $L[g(t)] = G(s)$,then $L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

Where a and b are constants.

Proof:

We know that $L[aF(s) \pm bG(s)] = aL[F(s)] \pm bL[G(s)]$

$$= a\,F(s) \pm b\,G(s)$$

$(i.e.)\,a\,F(s) \pm b\,G(s) = L[af(t) \pm bg(t)]$

Operating L^{-1} on both sides, we get

$L^{-1}[aF(s) \pm bG(s)] = af(t) \pm bg(t)$

$L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

$\because f(t) = L^{-1}[F(s)]$
$\because g(t) = L^{-1}[G(s)]$

Result: 2 First shifting property

(i) $L^{-1}\,[F(s+a) = e^{-at}L^{-1}[F(s)]]$

(ii) $L^{-1}\,[F(s-a) = e^{at}L^{-1}[F(s)]]$

Proof:

Let $L[e^{-at}f(t)] = F[s+a]$

Operating L^{-1} on both sides, we get

$e^{-at}f(t) = L^{-1}[F[s+a]]$

$L^{-1}[F[s+a]] = e^{-at}L^{-1}[F(s)]$

Result: 3 Multiplication by s.

If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$, then $L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) = sF(s)$

Operating L^{-1} on both sides, we get

$f'(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}f(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}L^{-1}[F(s)] = L^{-1}[sF(s)]$

$\therefore L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Result: 4 Division by s.

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]dt$$

Proof:

We know that $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)] = \frac{1}{s}F(s)$

Operating L^{-1} on both sides ,we get

$$\int_0^t f(t)dt = L^{-1}\left[\frac{1}{s}F(s)\right]$$

$$\int_0^t L^{-1}[F(s)]\,dt = L^{-1}\left[\frac{1}{s}F(s)\right]$$

$$\therefore L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$$

Result: 5 Inverse Laplace transform of derivative

$$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

Proof:

We know that $L[tf(t)] = \frac{-d}{ds}L[f(t)] = \frac{-d}{ds}F(s)$

Operating L^{-1} on both sides ,we get

$$tf(t) = -L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$f(t) = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

Result: 6 Inverse Laplace transform of integral

$$L^{-1}[F(s)] = tL^{-1}\left[\int_s^\infty F(s)ds\right]$$

Proof:

We know that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L(f(t))\,ds$

$$= \int_s^\infty F(s)\,ds$$

Operating L^{-1} on both sides, we get

$$\frac{f(t)}{t} = L^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

$$f(t) = tL^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

$$L^{-1}[F(s)] = tL^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

Problems under inverse Laplace transform of elementary functions

Example: 5.39 Find the inverse Laplace for the following

(i) $\frac{1}{2s+3}$ **(ii)** $\frac{1}{4s^2+9}$ **(iii)** $\frac{s^3-3s^2+7}{s^4}$ **(iv)** $\frac{3s+5}{s^2+36}$

Solution:

(i) $L^{-1}\left[\frac{1}{2s+3}\right]=L^{-1}\left[\frac{1}{2\left[s+\frac{3}{2}\right]}\right]$

$$=\frac{1}{2}e^{-\frac{3t}{2}}$$

(ii) $L^{-1}\left[\frac{1}{4s^2+9}\right]=L^{-1}\left[\frac{1}{4\left[s^2+\frac{9}{4}\right]}\right]$

$$=\frac{1}{4}L^{-1}\left[\frac{1}{\left[s^2+\frac{9}{4}\right]}\right]$$

$$=\frac{1}{4}\frac{1}{3/2}sin\frac{3}{2}t$$

$$=\frac{1}{6}sin\frac{3}{2}t$$

(iii) $L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right]=L^{-1}\left[\frac{s^3}{s^4}-\frac{3s^2}{s^4}+\frac{7}{s^4}\right]$

$$=L^{-1}\left[\frac{1}{s}\right]-3L^{-1}\left[\frac{1}{s^2}\right]+7L^{-1}\left[\frac{1}{s^4}\right]$$

$$L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right]=1-3t+\frac{7t^3}{3!}$$

(iv) $L^{-1}\left[\frac{3s+5}{s^2+36}\right]=3L^{-1}\left[\frac{s}{s^2+36}\right]+5L^{-1}\left[\frac{1}{s^2+36}\right]$

$$L^{-1}\left[\frac{3s+5}{s^2+36}\right]=3cos6t+\frac{5sin6t}{6}$$

Inverse Laplace transform using First shifting theorem

$$\boldsymbol{L^{-1}[F(s+a)]=e^{-at}L^{-1}[F(s)]}$$

Example: 5.40 Find the inverse Laplace transform for the following:

(i)	$\frac{1}{(s+2)^2}$	**(ii)**	$\frac{1}{(s-3)^4}$	**(iii)**	$\frac{1}{(s+3)^2+9}$	**(iv)**	$\frac{1}{s^2-2s+2}$
(v)	$\frac{1}{s^2-4s+13}$	**(vi)**	$\frac{s+2}{(s+2)^2+25}$	**(vii)**	$\frac{s+2}{s^2+4s+20}$	**(viii)**	$\frac{s}{(s+3)^2}$
(ix)	$\frac{s}{(s-4)^3}$	**(x)**	$\frac{s}{s^2-2s+2}$	**(xi)**	$\frac{2s+3}{s^2+6s+25}$	**(xii)**	$\frac{s}{s^2+6s-7}$

Solution:

(i) $L^{-1}\left[\frac{1}{(s+2)^2}\right]=e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]=e^{-2t}t$

(ii) $L^{-1}\left[\frac{1}{(s-3)^4}\right]=e^{3t}L^{-1}\left[\frac{1}{s^4}\right]=e^{-2t}\frac{t^3}{3!}$

(iii) $L^{-1}\left[\frac{1}{(s+3)^2+9}\right]=e^{-3t}L^{-1}\left[\frac{1}{s^2+9}\right]=e^{-3t}\frac{sin3t}{3}$

(iv) $L^{-1}\left[\frac{1}{s^2-2s+2}\right]=L^{-1}\left[\frac{1}{(s-1)^2+1}\right]=e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]=e^{t}sint$

(v) $L^{-1}\left[\frac{1}{s^2-4s+13}\right]=L^{-1}\left[\frac{1}{(s-2)^2+9}\right]=e^{2t}L^{-1}\left[\frac{1}{s^2+9}\right]=e^{2t}\frac{sin3t}{3}$

(vi) $L^{-1}\left[\frac{s+2}{(s+2)^2+25}\right]=e^{-2t}L^{-1}\left[\frac{s}{s^2+25}\right]=e^{-2t}cos5t$

(vii) $L^{-1}\left[\frac{s+2}{s^2+4s+20}\right]=L^{-1}\left[\frac{s+2}{(s+2)^2+16}\right]$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+16}\right] = e^{-2t}cos4t$$

(viii) $L^{-1}\left[\frac{s}{(s+3)^2}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2}\right]$

$$= L^{-1}\left[\frac{s+3}{(s+3)^2}\right] - L^{-1}\left[\frac{3}{(s+3)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s+3}\right] - 3L^{-1}\left[\frac{1}{(s+3)^2}\right]$$

$$= e^{-3t} - 3e^{-3t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-3t} - 3e^{-3t}t$$

(ix) $L^{-1}\left[\frac{s}{(s-4)^3}\right] = L^{-1}\left[\frac{s-4+4}{(s-4)^3}\right]$

$$= L^{-1}\left[\frac{s-4}{(s-4)^3}\right] + L^{-1}\left[\frac{4}{(s-4)^3}\right]$$

$$= L^{-1}\left[\frac{1}{(s-4)^2}\right] + 4L^{-1}\left[\frac{1}{(s-4)^3}\right]$$

$$= e^{4t}L^{-1}\left[\frac{1}{s^2}\right] + 4e^{4t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$= e^{4t}t + 4e^{4t}\frac{t^2}{2!}$$

$$= e^{4t}t + 2e^{4t}t^2$$

(x) $L^{-1}\left[\frac{s}{s^2-2s+2}\right] = L^{-1}\left[\frac{s}{(s-1)^2+1}\right] = L^{-1}\left[\frac{s-1+1}{(s-1)^2+1}\right]$

$$= L^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] + L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$= e^{t}L^{-1}\left[\frac{s}{s^2+1}\right] + e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$L^{-1}\left[\frac{s}{s^2-2s+2}\right] = e^{t}cost + e^{t}sint$$

(xi) $L^{-1}\left[\frac{2s+3}{s^2+6s+25}\right] = L^{-1}\left[\frac{2s+3}{(s+3)^2+16}\right] = L^{-1}\left[\frac{2(s+3-3)+3}{(s+3)^2+16}\right]$

$$= L^{-1}\left[\frac{2(s+3)-6+3}{(s+3)^2+16}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{2s-3}{s^2+16}\right]$$

$$= e^{-3t}\left[2L^{-1}\left[\frac{s}{s^2+16}\right] - 3L^{-1}\left[\frac{1}{s^2+16}\right]\right]$$

$$L^{-1}\left[\frac{2s+3}{s^2+6s+25}\right] = e^{-3t}\left(2cos4t - \frac{3sin4t}{4}\right)$$

(xii) $L^{-1}\left[\frac{s}{s^2+6s-7}\right] = L^{-1}\left[\frac{s}{(s+3)^2-16}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2-16}\right]$

$$= e^{-3t}L^{-1}\left[\frac{s-3}{s^2-16}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{s}{s^2-16}\right] - 3e^{-3t}L^{-1}\left[\frac{1}{s^2-16}\right]$$

$$L^{-1}\left[\frac{s}{s^2+6s-7}\right] = e^{-3t}\left[cosh4t - \frac{3sinh4t}{4}\right]$$

Inverse Using partial Fraction

Example: 5.47 Find $L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right]$

Solution:

$$\frac{2s-3}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$= \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2}$$

$$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 2s-3 \cdots (1)$$

Put $s = 1$ in (1)

$A = -1$

Put $s = 2$ in (1)

$C = 1$

Equating the coefficient of s^2

$A + B = 0$

$B = -A \Rightarrow B = 1$

$$\therefore \frac{2s-3}{(s-1)(s-2)^2} = \frac{-1}{s-1} + \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

$$L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right] = -L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s-2}\right] + \left[\frac{1}{(s-2)^2}\right]$$

$$= -e^{t} + e^{2t} + e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$\therefore L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right] = -e^{t} + e^{2t} + e^{2t}t$$

Example: 5.49 Find the inverse Laplace transform of $\frac{4s+5}{(s+1)(s^2+4)}$

Solution:

$$\frac{4s+5}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+c}{s^2+4}$$

$$= \frac{A(s^2+4)+(Bs+c)(s+1)}{(s+1)(s^2+4)}$$

$$A(s^2+4) + (Bs+c)(s+1) = 4s + 5 \cdots\cdots (1)$$

Put $s = -1$ in (1)

$A(1+4) + 0 = 4(-1) + 5$

$A(5) = 1 \Rightarrow A = \frac{1}{5}$

Equating coefficients of s^2 term in (1)

$A + B = 0$

$B = -A \Rightarrow B = \frac{-1}{5}$

Put $s = 0$ in (1)

$A(4) + C = 5$
$C = 5 - 4A = 5 - \frac{4}{5}$

$= \frac{25-4}{5} = \frac{21}{5}$

$$\therefore \frac{4s+5}{(s+1)(s^2+4)} = \frac{\frac{1}{5}}{s+1} + \frac{\frac{-1}{5}s + \frac{21}{5}}{s^2+4}$$

$$= \frac{1}{5(s+1)} - \frac{s}{5(s^2+4)} + \frac{21}{5}\frac{1}{(s^2+4)}$$

$$L^{-1}\left[\frac{4s+5}{(s+1)(s^2+4)}\right] = \frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{21}{5}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}cos2t + \frac{21}{5}\frac{sin2t}{2}$$

$$L^{-1}\left[\frac{4s+5}{(s+1)(s^2+4)}\right] = \frac{1}{5}e^{-t} - \frac{1}{5}cos2t + \frac{21}{10}sin2t$$

Example: 5.48 Find the inverse Laplace transform of $\frac{5s^2-15s-11}{(s+1)(s-2)^3}$

Solution:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}+\frac{C}{(s-2)^3}$$

$$=\frac{A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1)}{(s-1)(s-2)^3}$$

$$A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1)=5s^2-15s-11\cdots(1)$$

Put $s=-1$ in (1)

$A(-27)=9$

$A=\frac{9}{-27}\Rightarrow A=\frac{-1}{3}$

Put $s=2$ in (1)

$D(3)=-21$

$D=\frac{-21}{3}=-7$

Equating the coefficient of s^3

$A+B=0$

$B=-A\Rightarrow B=\frac{1}{3}$

Put $s=0$ in (1), we get

$-8A+4B-2C+D=-11$

$\frac{8}{3}+\frac{4}{3}-2C-7=-11$

$4-2C=7-11$

$-2C=-8\Rightarrow C=4$

$$\therefore\frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{-1}{3(s+1)}+\frac{1}{3(s-2)}+\frac{4}{(s-2)^2}-\frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=\frac{-1}{3}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2}\right]-7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$=\frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=\frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}\frac{t^2}{2}$$

Week 5

Topics: Initial and Boundary Value Problem

Pages (31-50)

Solving Ordinary Differential Equation

Problem:
$Y'' + aY' + bY = G(t)$ subject to the initial conditions $Y(0) = A$, $Y'(0) = B$ where a, b, A, B are constants.

Solution:

- Laplace transform of Y(t) be y(s), or, more concisely, y.
- Then solve for y in terms of s.
- Take the inverse transform, we obtain the desired solution Y.

5.9 SOLUTION OF DIFFERENTIAL EQUATION BY LAPLACE TRANSFORM TECHNIQUE

There are so many methods to solve a linear differential equation. If the initial conditions are known, then Laplace transform technique is easier to solve the differential equation. The Laplace transform transforms the differential equation into an algebraic equation.

$$L[y'(t)] = sL[y(t)] - y(0)$$
$$L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$$

Problems using Partial Fraction

Example: 5.66 Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x = 0$ and $\frac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method.

Solution:

Given $x'' - 3x' + 2x = 2; x(0) = 0; x'(0) = 5$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 3L[x'(t)] + 2L[x(t)] = 2L(1)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 3[sL[x(t)] - x(0)] + 2L[x(t)] = \frac{2}{s}$$

Substituting $x(0) = 0; x'(0) = 5$

$$[s^2L[x(t)] - 0 - 5] - 3[sL[x(t)] - 0] + 2L[x(t)] = \frac{2}{s}$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 3s\bar{x} + 2\bar{x} = \frac{2}{s} + 5$$

$$[s^2 - 3s + 2]\bar{x} = \frac{2}{s} + 5$$

$$(s-1)(s-2)\bar{x} = \frac{2}{s} + 5$$

$$\bar{x} = \frac{2+5s}{s(s-1)(s-2)}$$

Consider $\frac{2+5s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{A(s-1)(s-2)+Bs(s-2)+Cs(s-1)}{s(s-1)(s-2)}$$

$$A(s-1)(s-2) + Bs(s-2) + Cs(s-1) = 2 + 5s \cdots (1)$$

Put $s = 0$ in (1)	Put $s = 1$ in (1)	Put $s = 2$ in (1)
$A(-1)(-2) = 2$	$B(1)(-1) = 7$	$C(2)(1) = 2 + 10$
$A = 1$	$B = -7$	$C = 6$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}$$

$$\therefore \bar{x} = \frac{1}{s} - 7\frac{1}{s-1} + 6\frac{1}{s-2}$$

$$x(t) = L^{-1}\left[\frac{1}{s}\right] - 7L^{-1}\left[\frac{1}{s-1}\right] + 6L^{-1}\left[\frac{1}{s-2}\right]$$

$$x(t) = 1 - 7e^{t} + 6e^{2t}$$

Example: 5.67 Using Laplace transform solve the differential equation $y'' - 3y' - 4y = 2e^{-t}$, **with** $y(0) = 1 = y'(0)$.

Solution:

Given $y'' - 3y' - 4y = 2e^{-t}$; with $y(0) = 1 = y'(0)$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 3L[y'(t)] - 4L[y(t)] = 2L(e^{-t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] - 4L[y(t)] = 2\frac{1}{s+1}$$

Substituting $y(0) = 1 = y'(0)$.

$$[s^2L[y(t)] - s - 1] - 3[sL[y(t)] - 1] - 4L[y(t)] = \frac{2}{s+1}$$

$$s^2L[y(t)] - s - 1 - 3sL[y(t)] + 3 - 4L[y(t)] = \frac{2}{s+1}$$

$$s^2L[y(t)] - 3sL[y(t)] - 4L[y(t)] = \frac{2}{s+1} + s - 2$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} - 3s\bar{y} - 4\bar{y} = \frac{2}{s+1} + s - 2$$

$$[s^2 - 3s - 4]\bar{y} = \frac{2}{s+1} + s - 2$$

$$[s^2 - 3s - 4]\bar{y} = \frac{2 + s(s+1) - 2(s+1)}{s+1}$$

$$= \frac{2 + s^2 + s - 2s - 2}{s+1}$$

$$(s+1)(s-4)\bar{y} = \frac{s^2 - s}{s+1}$$

$$\bar{y} = \frac{s^2 - s}{(s+1)(s+1)(s-4)}$$

$$\bar{y} = \frac{s^2 - s}{(s+1)^2(s-4)}$$

Consider $\frac{s^2 - s}{(s+1)^2(s-4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}$

$$\frac{s^2 - s}{(s+1)^2(s-4)} = \frac{A(s+1)(s-4) + B(s-4) + C(s+1)^2}{(s+1)^2(s-4)}$$

$$A(s+1)(s-4) + B(s-4) + C(s+1)^2 = s^2 - s \cdots (1)$$

Put $s = -1$ in (1)	Put $s = 4$ in (1)	equating the coefficients of s^2, we get
$-5B = 1 + 1$	$25C = 16 - 4$	$A + C = 1 \Rightarrow A = 1 - C \Rightarrow 1 - \frac{12}{25}$
$B = \frac{-2}{5}$	$C = \frac{12}{25}$	$A = \frac{13}{25}$

$$\frac{s^2 - s}{(s+1)^2(s-4)} = \frac{25}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$$

$$\therefore \bar{y} = \frac{13}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$$

$$y(t) = \frac{13}{25}L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{2}{5}L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{12}{25}L^{-1}\left[\frac{1}{s-4}\right]$$

$$y(t) = \frac{13}{25}e^{-t} - \frac{2}{5}te^{-t} + \frac{12}{25}e^{4t}$$

Example: 5.68 Solve the differential equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}$, with $y(0) = 1$ and $y'(0) = 0$ using Laplace transform.

Solution:

Given $y'' - 3y' + 2y = e^{-t}$; with $y(0) = 1$ and $y'(0) = 1$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(e^{-t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$$

Substituting $y(0) = 1$ and $y'(0) = 0$.

$$[s^2L[y(t)] - s - 0] - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{1}{s+1}$$

$$s^2L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$$

$$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{1}{s+1} + s - 3$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{1}{s+1} + s - 3$$

$$[s^2 - 3s + 2]\bar{y} = \frac{1}{s+1} + s - 3$$

$$[s^2 - 3s + 2]\bar{y} = \frac{1+s(s+1)-3(s+1)}{s+1}$$

$$= \frac{1+s^2+s-3s-3}{s+1}$$

$$(s-1)(s-2)\bar{y} = \frac{s^2-2s-2}{s+1}$$

$$\bar{y} = \frac{s^2-2s-2}{(s+1)(s-1)(s-2)}$$

Consider $\frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$

$$\frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{A(s-1)(s-2)+B(s+1)(s-2)+C(s+1)(s-1)}{(s+1)(s-1)(s-2)}$$

$$A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1) = s^2 - 2s - 2 \cdots (1)$$

Put $s = -1$ in (1) $6A = 1 + 2 - 2$ $A = \frac{1}{6}$	put $s = 1$ in (1) $-2B = 1 - 4$ $B = \frac{3}{2}$	put $s = 2$ in (1) $3C = 4 - 4 - 2$ $C = \frac{-2}{3}$

$$\therefore \frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$$

$$\bar{y} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$$

$y(t) = \frac{1}{6}L^{-1}\left[\frac{1}{(s+1)}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$

$y(t) = \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} - \frac{2}{3}e^{2t}$

Example: 5.69 Using Laplace transform solve the differential equation $y'' + 2y' - 3y = sint$, with $y(0) = y'(0) = 0$.

Solution:

Given $y'' + 2y' - 3y = sint$ with $y(0) = 0 = y'(0)$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] + 2L[y'(t)] - 3L[y(t)] = L(sint)$

$[s^2L[y(t)] - sy(0) - y'(0)] + 2[sL[y(t)] - y(0)] - 3L[y(t)] = \frac{1}{s^2+1}$

Substituting $y(0) = 0 = y'(0)$.

$[s^2L[y(t)] - 0 - 0] + 2[sL[y(t)] - 0] - 3L[y(t)] = \frac{1}{s^2+1}$

$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$

$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}$

$[s^2 + 2s - 3]\bar{y} = \frac{1}{s^2+1}$

$(s-1)(s+3)\bar{y} = \frac{1}{s^2+1}$

$\bar{y} = \frac{1}{(s-1)(s+3)(s^2+1)}$

Consider $\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$

$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A(s^2+1)(s+3)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)}{(s-1)(s+3)(s^2+1)}$

$A(s^2+1)(s+3) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) = 1 \cdots (1)$

Put $s = 1$ in (1)	Put $s = -3$ in (1)	equating the coefficients of s^2, we get
$8A = 0 + 1$	$B(-4)(10) = 1$	$A + B + C = 0 \Rightarrow C = -A - B = \frac{-1}{8} + \frac{1}{40}$
$A = \frac{1}{8}$	$B = \frac{-1}{40}$	$C = \frac{-1}{10}$

Put $s = 0$ in (1), we get

$3A - B - 3D = 1 \Rightarrow \frac{3}{8} + \frac{1}{40} - 3D = 1$

$3D = \frac{3}{8} + \frac{1}{40} - 1$

$3D = \frac{15+1-40}{40} \Rightarrow D = \frac{-24}{40\times3} \Rightarrow D = \frac{-1}{5}$

$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} + \frac{(\frac{-1}{10})s - \frac{1}{5}}{s^2+1}$

$$\therefore \bar{y} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} - \frac{s}{10(s^2+1)} - \frac{1}{5(s^2+1)}$$

$$y(t) = \frac{1}{8}L^{-1}\left[\frac{1}{(s-1)}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(cost - 2sint)$$

Example: 5.70 Using Laplace transform solve the differential equation $y'' - 3y' + 2y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$.

Solution:

Given $y'' - 3y' + 2y = 4e^{2t}$; with $y(0) = -3$ and $y'(0) = 5$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = 4L(e^{2t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = 4\frac{1}{s-2}$$

Substituting $y(0) = -3$ and $y'(0) = 5$.

$$[s^2L[y(t)] + 3s - 5] - 3[sL[y(t)] + 3] + 2L[y(t)] = \frac{4}{s-2}$$

$$s^2L[y(t)] + 3s - 5 - 3sL[y(t)] - 9 + 2L[y(t)] = \frac{4}{s-2}$$

$$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{4}{s-2} - 3s + 14$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{4}{s-2} - 3s + 14$$

$$[s^2 - 3s + 2]\bar{y} = \frac{4}{s-2} + 14 - 3s$$

$$[s^2 - 3s + 2]\bar{y} = \frac{4 + (14-3s)(s-2)}{s-2}$$

$$(s-1)(s-2)\,\bar{y} = \frac{4 + (14-3s)(s-2)}{s-2}$$

$$\bar{y} = \frac{4 + (14-3s)(s-2)}{(s-1)(s-2)^2}$$

Consider $\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

$$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2}$$

$$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 4 + (14-3s)(s-2) \cdots (1)$$

Put $s = 1$ in (1)	Put $s = 2$ in (1)	equating the coefficients of s^2, we get
$A = 4 - 11$	$C = 4 + 0$	$A + B = -3 \Rightarrow -7 + B = -3$
$A = -7$	$C = 4$	$B = 4$

$$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\therefore \bar{y} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$y(t) = -7L^{-1}\left[\frac{1}{(s-1)}\right] + 4L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$= -7e^{t} + 4e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$

$y(t) = -7e^{t} + 4e^{2t} + 4e^{2t}t$

Example: 5.71 Using Laplace transform solve the differential equation $y'' - 4y' + 8y = e^{2t}$, with $y(0) = 2$ and $y'(0) = -2$.

Solution:

Given $y'' - 4y' + 8y = e^{2t}$; with $y(0) = 2$ and $y'(0) = -2$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 4L[y'(t)] + 8L[y(t)] = L(e^{2t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 4[sL[y(t)] - y(0)] + 8L[y(t)] = \frac{1}{s-2}$

Substituting $y(0) = 2$ and $y'(0) = -2$.

$[s^2L[y(t)] - 2s + 2] - 4[sL[y(t)] - 2] + 8L[y(t)] = \frac{1}{s-2}$

$s^2L[y(t)] - 2s + 2 - 4sL[y(t)] + 8 + 8L[y(t)] = \frac{1}{s-2}$

$s^2L[y(t)] - 4sL[y(t)] + 8L[y(t)] = \frac{1}{s-2} + 2s - 10$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 4s\bar{y} + 8\bar{y} = \frac{1}{s-2} + 2s - 10$

$[s^2 - 4s + 8]\bar{y} = \frac{1}{s-2} + 2s - 10$

$[s^2 - 4s + 8]\bar{y} = \frac{1+(2s-10)(s-2)}{s-2}$

$\bar{y} = \frac{1+(2s-10)(s-2)}{(s-2)(s^2-4s+8)}$

$= \frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]}$

Consider $\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{A}{s-2} + \frac{B(s-2)+C}{(s-2)^2+4}$

$= \frac{A[(s-2)^2+4]+B[(s-2)+C](s-2)}{[s-2][(s-2)^2+4]}$

$A[(s-2)^2 + 4] + B[(s-2) + C](s-2) = 1 + (2s-10)(s-2) \cdots (1)$

Put $s = 2$ in (1)	Put $s = 0$ in (1)	equating the coefficients of s^2, we get
$4A = 1 + 0$	$8A + 4B - 2C = 21$	$A + B = 2 \Rightarrow \frac{1}{4} + B =$ 2
$A = \frac{1}{4}$	$C = -6$	$B = \frac{7}{4}$

$\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$

$\therefore \bar{y} = \frac{1}{4(s-2)} + \frac{7}{4}\frac{(s-2)}{(s-2)^2+4} - 6\frac{1}{(s-2)^2+4}$

$y(t) = \frac{1}{4}L^{-1}\left[\frac{1}{(s-2)}\right] + \frac{7}{4}L^{-1}\left[\frac{(s-2)}{(s-2)^2+4}\right] - 6L^{-1}\left[\frac{1}{(s-2)^2+4}\right]$

$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}L^{-1}\left[\frac{s}{s^2+4}\right] - 6e^{2t}L^{-1}\left[\frac{1}{s^2+4}\right]$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}cos2t - 6e^{2t}\frac{sin2t}{2}$$

$$y(t) = \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}cos2t - 3e^{2t}sin2t$$

Problems without using Partial Fraction

Example: 5.72 Solve using Laplace transform $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, with $x = 2, \frac{dx}{dt} = -1$ at $t = 0$

Solution:

Given $x'' - 2x' + x = e^t; x(0) = 2; x'(0) = -1$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L(e^t)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 2[sL[x(t)] - x(0)] + L[x(t)] = \frac{1}{s-1}$$

Substituting $x(0) = 2; x'(0) = -1$

$$[s^2L[x(t)] - 2s + 1] - 2[sL[x(t)] - 2] + L[x(t)] = \frac{1}{s-1}$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 2s\bar{x} + \bar{x} = \frac{1}{s-1} + 2s - 5$$

$$[s^2 - 2s + 1]\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$(s-1)^2\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$\bar{x} = \frac{1}{(s-1)(s-1)^2} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$$

$$x(t) = L^{-1}\left[\frac{1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - 5L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^tL^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - 5e^tL^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^t\frac{t^2}{2!} + 2L^{-1}\left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2!} + 2L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2!} + 2e^t + 2e^tL^{-1}\left[\frac{1}{s^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2} + 2e^t + 2e^tt - 5e^tt$$

$$\therefore x = \frac{t^2e^t}{2} + 2e^t - 3e^tt$$

Exercise: Find the Inverse Laplace transformation of the followings:

$(i)\ \dfrac{6s-10}{s^2-4s+20}$ $\qquad$ $(i)\ \dfrac{4s+12}{s^2+8s+16}$

$(iii)\ \dfrac{3s-8}{4s^2-25}$ $\qquad$ $(iv)\ \dfrac{2s+4}{s^2+2s+5}$

The Laplace transform is a well established mathematical technique for solving a differential equation. Many mathematical problems are solved using transformations. The idea is to transform the problem into another problem that is easier to solve. On the other side, the inverse transform is helpful to calculate the solution to the given problem.

For better understanding, let us solve a first-order differential equation with the help of Laplace transformation,

Consider y′- 2y = e^{3x} and y(0) = -5. Find the value of L(y).

First step of the equation can be solved with the help of the linearity equation:

L(y′ – 2y] = L(e^{3x})

L(y′) – L(2y) = 1/(s-3)

(because L(e^{ax}) = 1/(s-a))

L(y′) – 2s(y) = 1/(s-3)

sL(y) – y(0) – 2L(y) = 1/(s-3)

(Using Linearity property of the Laplace transform)

L(y)(s-2) + 5 = 1/(s-3) (Use value of y(0) ie -5 (given))

L(y)(s-2) = 1/(s-3) – 5

L(y) = (-5s+16)/(s-2)(s-3) …..(1)

here (-5s+16)/(s-2)(s-3) can be written as -6/s-2 + 1/(s-3) using partial fraction method

(1) implies L(y) = -6/(s-2) + 1/(s-3)

L(y) = -6e^{2x} + e^{3x}

Question: Solve $Y' + Y = t, \quad Y(0) = 1, Y'(0) = -2.$

Solution: Given that, $Y'' + Y = t$

Taking the Laplace transformation on both sides of the differential equation and using the given conditions, we have $\mathcal{L}\{Y''\} + \mathcal{L}\{Y'\} = \mathcal{L}\{t\}$

$$\Rightarrow s^2 y - sY(0) - Y'(0) + y = \frac{1}{s^2}$$

$$\Rightarrow s^2 y - s + 2 + y = \frac{1}{s^2}$$

$$\Rightarrow y = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}$$

$$= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

$$= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$$

Again taking Inverse Laplace Transformations, we have

$$\mathcal{L}^{-1}\{y\} = Y = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right\}$$

$$Y = t + cost - 3sint$$

Question: Solve the following equations using Laplace transformations

$Y'(t) + 4Y(t) = 12t, \quad Y(0) = 0, \quad Y'(0) = 7$

$Y'(t) - 3Y'(t) + 2Y(t) = 4e^{2t}, \quad Y(0) = -3, \quad Y'(0) = 5$

Solve $Y'' - 3Y' + 2Y = 4e^{2t}$, $Y(0) = -3$, $Y'(0) = 5$.

We have

$$\mathcal{L}\{Y''\} - 3\mathcal{L}\{Y'\} + 2\mathcal{L}\{Y\} = 4\mathcal{L}\{e^{2t}\}$$

$$\{s^2y - sY(0) - Y'(0)\} - 3\{sy - Y(0)\} + 2y = \frac{4}{s-2}$$

$$\{s^2y + 3s - 5\} - 3\{sy + 3\} + 2y = \frac{4}{s-2}$$

$$(s^2 - 3s + 2)y + 3s - 14 = \frac{4}{s-2}$$

$$y = \frac{4}{(s^2-3s+2)(s-2)} + \frac{14-3s}{s^2-3s+2}$$

$$= \frac{-3s^2+20s-24}{(s-1)(s-2)^2}$$

$$= \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

Thus

$$Y = \mathcal{L}^{-1}\left\{\frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}\right\} = -7e^t + 4e^{2t} + 4te^{2t}$$

which can be verified as the solution.

5.9 SOLUTION OF DIFFERENTIAL EQUATION BY LAPLACE TRANSFORM TECHNIQUE

There are so many methods to solve a linear differential equation. If the initial conditions are known, then Laplace transform technique is easier to solve the differential equation. The Laplace transform transforms the differential equation into an algebraic equation.

$$L[y'(t)] = sL[y(t)] - y(0)$$
$$L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$$

Problems using Partial Fraction

Example: 5.66 Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x = 0$ and $\frac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method.

Solution:

Given $x'' - 3x' + 2x = 2;\ x(0) = 0;\ x'(0) = 5$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 3L[x'(t)] + 2L[x(t)] = 2L(1)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 3[sL[x(t)] - x(0)] + 2L[x(t)] = \frac{2}{s}$$

Substituting $x(0) = 0;\ x'(0) = 5$

$$[s^2L[x(t)] - 0 - 5] - 3[sL[x(t)] - 0] + 2L[x(t)] = \frac{2}{s}$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 3s\bar{x} + 2\bar{x} = \frac{2}{s} + 5$$

$$[s^2 - 3s + 2]\bar{x} = \frac{2}{s} + 5$$

$$(s-1)(s-2)\bar{x} = \frac{2}{s} + 5$$

$$\bar{x} = \frac{2+5s}{s(s-1)(s-2)}$$

Consider $\frac{2+5s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{A(s-1)(s-2) + Bs(s-2) + Cs(s-1)}{s(s-1)(s-2)}$$

$$A(s-1)(s-2) + Bs(s-2) + Cs(s-1) = 2 + 5s \cdots (1)$$

Put $s = 0$ in (1)	Put $s = 1$ in (1)	Put $s = 2$ in (1)
$A(-1)(-2) = 2$	$B(1)(-1) = 7$	$C(2)(1) = 2 + 10$
$A = 1$	$B = -7$	$C = 6$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}$$

$\therefore \bar{x} = \frac{1}{s} - 7\frac{1}{s-1} + 6\frac{1}{s-2}$

$x(t) = L^{-1}\left[\frac{1}{s}\right] - 7L^{-1}\left[\frac{1}{s-1}\right] + 6L^{-1}\left[\frac{1}{s-2}\right]$

$x(t) = 1 - 7e^{t} + 6e^{2t}$

Example: 5.67 Using Laplace transform solve the differential equation $y'' - 3y' - 4y = 2e^{-t}$**, with** $y(0) = 1 = y'(0)$.

Solution:

Given $y'' - 3y' - 4y = 2e^{-t}$; with $y(0) = 1 = y'(0)$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] - 4L[y(t)] = 2L(e^{-t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] - 4L[y(t)] = 2\frac{1}{s+1}$

Substituting $y(0) = 1 = y'(0)$.

$[s^2L[y(t)] - s - 1] - 3[sL[y(t)] - 1] - 4L[y(t)] = \frac{2}{s+1}$

$s^2L[y(t)] - s - 1 - 3sL[y(t)] + 3 - 4L[y(t)] = \frac{2}{s+1}$

$s^2L[y(t)] - 3sL[y(t)] - 4L[y(t)] = \frac{2}{s+1} + s - 2$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} - 4\bar{y} = \frac{2}{s+1} + s - 2$

$[s^2 - 3s - 4]\bar{y} = \frac{2}{s+1} + s - 2$

$[s^2 - 3s - 4]\bar{y} = \frac{2+s(s+1)-2(s+1)}{s+1}$

$= \frac{2+s^2+s-2s-2}{s+1}$

$(s+1)(s-4)\bar{y} = \frac{s^2-s}{s+1}$

$\bar{y} = \frac{s^2-s}{(s+1)(s+1)(s-4)}$

$\bar{y} = \frac{s^2-s}{(s+1)^2(s-4)}$

Consider $\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}$

$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A(s+1)(s-4)+B(s-4)+C(s+1)^2}{(s+1)^2(s-4)}$

$A(s+1)(s-4) + B(s-4) + C(s+1)^2 = s^2 - s \cdots (1)$

Put $s = -1$ in (1)	Put $s = 4$ in (1)	equating the coefficients of s^2, we get
$-5B = 1 + 1$	$25C = 16 - 4$	$A + C = 1 \Rightarrow A = 1 - C \Rightarrow 1 - \frac{12}{25}$
$B = \frac{-2}{5}$	$C = \frac{12}{25}$	$A = \frac{13}{25}$

$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{25}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$

$\therefore \bar{y} = \frac{13}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$

$y(t) = \frac{13}{25} L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{2}{5} L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{12}{25} L^{-1}\left[\frac{1}{s-4}\right]$

$y(t) = \frac{13}{25} e^{-t} - \frac{2}{5} t e^{-t} + \frac{12}{25} e^{4t}$

Example: 5.68 Solve the differential equation $\frac{d^2 y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}$**, with** $y(0) = 1$ *and* $y'(0) = 0$ **using Laplace transform.**

Solution:

Given $y'' - 3y' + 2y = e^{-t}$; with $y(0) = 1$ *and* $y'(0) = 1$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(e^{-t})$

$[s^2 L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$

Substituting $y(0) = 1$ *and* $y'(0) = 0$.

$[s^2 L[y(t)] - s - 0] - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{1}{s+1}$

$s^2 L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$

$s^2 L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{1}{s+1} + s - 3$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{1}{s+1} + s - 3$

$[s^2 - 3s + 2]\bar{y} = \frac{1}{s+1} + s - 3$

$[s^2 - 3s + 2]\bar{y} = \frac{1 + s(s+1) - 3(s+1)}{s+1}$

$= \frac{1 + s^2 + s - 3s - 3}{s+1}$

$(s-1)(s-2)\bar{y} = \frac{s^2 - 2s - 2}{s+1}$

$\bar{y} = \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)}$

Consider $\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$

$\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)}{(s+1)(s-1)(s-2)}$

$A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1) = s^2 - 2s - 2 \cdots (1)$

Put $s = -1$ in (1)	put $s = 1$ *in* (1)	put $s = 2$ *in* (1)
$6A = 1 + 2 - 2$	$-2B = 1 - 4$	$3C = 4 - 4 - 2$
$A = \frac{1}{6}$	$B = \frac{3}{2}$	$C = \frac{-2}{3}$

$\therefore \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$

$\bar{y} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$

$$y(t) = \frac{1}{6}L^{-1}\left[\frac{1}{(s+1)}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$$

$$y(t) = \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} - \frac{2}{3}e^{2t}$$

Example: 5.69 Using Laplace transform solve the differential equation $y'' + 2y' - 3y = sint$, with $y(0) = y'(0) = 0$.

Solution:

Given $y'' + 2y' - 3y = sint$ with $y(0) = 0 = y'(0)$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] + 2L[y'(t)] - 3L[y(t)] = L(sint)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 2[sL[y(t)] - y(0)] - 3L[y(t)] = \frac{1}{s^2+1}$$

Substituting $y(0) = 0 = y'(0)$.

$$[s^2L[y(t)] - 0 - 0] + 2[sL[y(t)] - 0] - 3L[y(t)] = \frac{1}{s^2+1}$$

$$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$$

$$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}$$

$$[s^2 + 2s - 3]\bar{y} = \frac{1}{s^2+1}$$

$$(s-1)(s+3)\bar{y} = \frac{1}{s^2+1}$$

$$\bar{y} = \frac{1}{(s-1)(s+3)(s^2+1)}$$

Consider $\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A(s^2+1)(s+3)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)}{(s-1)(s+3)(s^2+1)}$$

$$A(s^2+1)(s+3) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) = 1 \cdots (1)$$

Put $s = 1$ in (1)	Put $s = -3$ in (1)	equating the coefficients of s^2, we get
$8A = 0 + 1$	$B(-4)(10) = 1$	$A + B + C = 0 \Rightarrow C = -A - B = \frac{-1}{8} + \frac{1}{40}$
$A = \frac{1}{8}$	$B = \frac{-1}{40}$	$C = \frac{-1}{10}$

Put $s = 0$ in (1), we get

$$3A - B - 3D = 1 \Rightarrow \frac{3}{8} + \frac{1}{40} - 3D = 1$$

$$3D = \frac{3}{8} + \frac{1}{40} - 1$$

$$3D = \frac{15+1-40}{40} \Rightarrow D = \frac{-24}{40\times3} \Rightarrow D = \frac{-1}{5}$$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} + \frac{(\frac{-1}{10})s - \frac{1}{5}}{s^2+1}$$

$$\therefore \bar{y} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} - \frac{s}{10(s^2+1)} - \frac{1}{5(s^2+1)}$$

$$y(t) = \frac{1}{8}L^{-1}\left[\frac{1}{(s-1)}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(cost - 2sint)$$

Example: 5.70 Using Laplace transform solve the differential equation $y'' - 3y' + 2y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$.

Solution:

Given $y'' - 3y' + 2y = 4e^{2t}$; with $y(0) = -3$ and $y'(0) = 5$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = 4L(e^{2t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = 4\frac{1}{s-2}$$

Substituting $y(0) = -3$ and $y'(0) = 5$.

$$[s^2L[y(t)] + 3s - 5] - 3[sL[y(t)] + 3] + 2L[y(t)] = \frac{4}{s-2}$$

$$s^2L[y(t)] + 3s - 5 - 3sL[y(t)] - 9 + 2L[y(t)] = \frac{4}{s-2}$$

$$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{4}{s-2} - 3s + 14$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{4}{s-2} - 3s + 14$$

$$[s^2 - 3s + 2]\bar{y} = \frac{4}{s-2} + 14 - 3s$$

$$[s^2 - 3s + 2]\bar{y} = \frac{4 + (14-3s)(s-2)}{s-2}$$

$$(s-1)(s-2)\,\bar{y} = \frac{4 + (14-3s)(s-2)}{s-2}$$

$$\bar{y} = \frac{4 + (14-3s)(s-2)}{(s-1)(s-2)^2}$$

Consider $\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

$$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2}$$

$$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 4 + (14-3s)(s-2) \cdots (1)$$

Put $s = 1$ in (1)

$A = 4 - 11$

$A = -7$

Put $s = 2$ in (1)

$C = 4 + 0$

$C = 4$

equating the coefficients of s^2, we get

$A + B = -3 \Rightarrow -7 + B = -3$

$B = 4$

$$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\therefore \bar{y} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$y(t) = -7L^{-1}\left[\frac{1}{(s-1)}\right] + 4L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$= -7e^t + 4e^{2t} + 4e^{2t}L^{-1}[\frac{1}{s^2}]$

$y(t) = -7e^t + 4e^{2t} + 4e^{2t}t$

Example: 5.71 Using Laplace transform solve the differential equation $y'' - 4y' + 8y = e^{2t}$, with $y(0) = 2$ and $y'(0) = -2$.

Solution:

Given $y'' - 4y' + 8y = e^{2t}$; with $y(0) = 2$ and $y'(0) = -2$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 4L[y'(t)] + 8L[y(t)] = L(e^{2t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 4[sL[y(t)] - y(0)] + 8L[y(t)] = \frac{1}{s-2}$

Substituting $y(0) = 2$ and $y'(0) = -2$.

$[s^2L[y(t)] - 2s + 2] - 4[sL[y(t)] - 2] + 8L[y(t)] = \frac{1}{s-2}$

$s^2L[y(t)] - 2s + 2 - 4sL[y(t)] + 8 + 8L[y(t)] = \frac{1}{s-2}$

$s^2L[y(t)] - 4sL[y(t)] + 8L[y(t)] = \frac{1}{s-2} + 2s - 10$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 4s\bar{y} + 8\bar{y} = \frac{1}{s-2} + 2s - 10$

$[s^2 - 4s + 8]\bar{y} = \frac{1}{s-2} + 2s - 10$

$[s^2 - 4s + 8]\bar{y} = \frac{1+(2s-10)(s-2)}{s-2}$

$\bar{y} = \frac{1+(2s-10)(s-2)}{(s-2)(s^2-4s+8)}$

$= \frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]}$

Consider $\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{A}{s-2} + \frac{B(s-2)+C}{(s-2)^2+4}$

$= \frac{A[(s-2)^2+4]+B[(s-2)+C](s-2)}{[s-2][(s-2)^2+4]}$

$A[(s-2)^2 + 4] + B[(s-2) + C](s-2) = 1 + (2s-10)(s-2) \cdots (1)$

Put $s = 2$ in (1)	Put $s = 0$ in (1)	equating the coefficients of s^2, we get
$4A = 1 + 0$	$8A + 4B - 2C = 21$	$A + B = 2 \Rightarrow \frac{1}{4} + B = 2$
$A = \frac{1}{4}$	$C = -6$	$B = \frac{7}{4}$

$\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$

$\therefore \bar{y} = \frac{1}{4(s-2)} + \frac{7}{4}\frac{(s-2)}{(s-2)^2+4} - 6\frac{1}{(s-2)^2+4}$

$y(t) = \frac{1}{4}L^{-1}[\frac{1}{(s-2)}] + \frac{7}{4}L^{-1}[\frac{(s-2)}{(s-2)^2+4}] - 6L^{-1}[\frac{1}{(s-2)^2+4}]$

$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}L^{-1}[\frac{s}{s^2+4}] - 6e^{2t}L^{-1}[\frac{1}{s^2+4}]$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}\cos 2t - 6e^{2t}\frac{\sin 2t}{2}$$

$$y(t) = \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}\cos 2t - 3e^{2t}\sin 2t$$

Problems without using Partial Fraction

Example: 5.72 Solve using Laplace transform $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, with $x = 2, \frac{dx}{dt} = -1$ at $t = 0$

Solution:

Given $x'' - 2x' + x = e^t; x(0) = 2; x'(0) = -1$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L(e^t)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 2[sL[x(t)] - x(0)] + L[x(t)] = \frac{1}{s-1}$$

Substituting $x(0) = 2; x'(0) = -1$

$$[s^2L[x(t)] - 2s + 1] - 2[sL[x(t)] - 2] + L[x(t)] = \frac{1}{s-1}$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 2s\bar{x} + \bar{x} = \frac{1}{s-1} + 2s - 5$$

$$[s^2 - 2s + 1]\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$(s-1)^2\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$\bar{x} = \frac{1}{(s-1)(s-1)^2} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$$

$$x(t) = L^{-1}\left[\frac{1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - 5L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^tL^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - 5e^tL^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^t\frac{t^2}{2!} + 2L^{-1}\left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2!} + 2L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2!} + 2e^t + 2e^tL^{-1}\left[\frac{1}{s^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2} + 2e^t + 2e^tt - 5e^tt$$

$$\therefore x = \frac{t^2e^t}{2} + 2e^t - 3e^tt$$

Week 6
Topics: L-R Circuit problem
Pages (51-56)

APPLICATIONS TO ELECTRICAL CIRCUITS

A simple electrical circuit [Fig. 3-2] consists of the following *circuit elements* connected in *series* with a *switch* or *key* K:

1. a *generator* or *battery*, supplying an *electromotive force* or *e.m.f.* E (volts),
2. a *resistor* having *resistance* R (ohms),
3. an *inductor* having *inductance* L (henrys),
4. a *capacitor* having *capacitance* C (farads).

These circuit elements are represented symbolically as in Fig. 3-2.

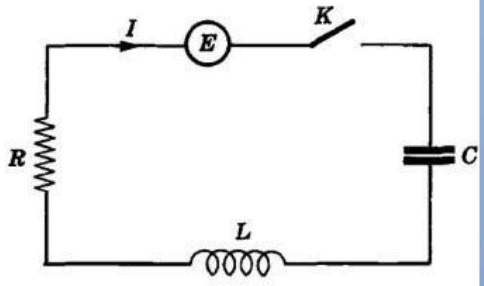


Fig. 3-2

When the switch or key K is closed, so that the circuit is completed, a charge Q (coulombs) will flow to the capacitor plates. The time rate of flow of charge, given by $\frac{dQ}{dt} = I$, is called the *current* and is measured in amperes when time t is measured in seconds.

More complex electrical circuits, as shown for example in Fig. 3-3, can occur in practice.

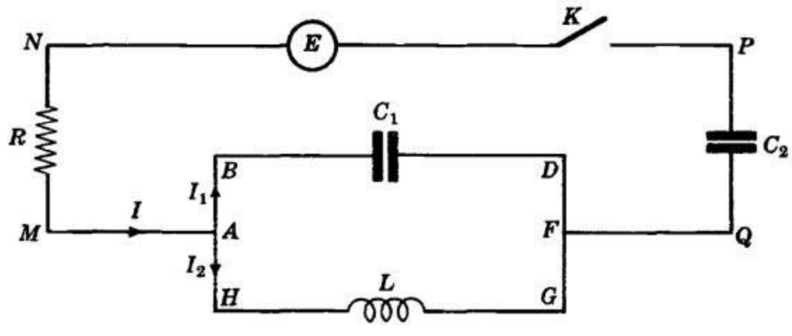


Fig. 3-3

An important problem is to determine the charges on the capacitors and currents as functions of time. To do this we define the *potential drop* or *voltage drop* across a circuit element.

(*a*) Voltage drop across a resistor $= RI = R\frac{dQ}{dt}$

(*b*) Voltage drop across an inductor $= L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$

(*c*) Voltage drop across a capacitor $= \frac{Q}{C}$

(*d*) Voltage drop across a generator $= -\text{Voltage rise} = -E$

The differential equations can then be found by using the following laws due to Kirchhoff.

Kirchhoff's Laws

1. The algebraic sum of the currents flowing toward any junction point [for example A in Fig. 3-3] is equal to zero.

2. The algebraic sum of the potential drops, or voltage drops, around any closed loop [such as $ABDFGHA$ or $ABDFQPNMA$ in Fig. 3-3] is equal to zero.

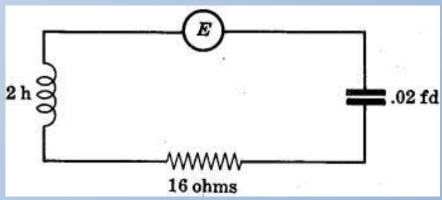
For the simple circuit of Fig. 3-2 application of these laws is particularly easy [the first law is actually not necessary in this case]. We find that the equation for determination of Q is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E \qquad (8)$$

By applying the laws to the circuit of Fig. 3-3, two simultaneous equations are obtained

Question: An inductor of 2 henrys, a resistor of 16 ohms and a capacitor of 0.02 farads are connected in series with an electromotive force (e.m.f) of 300 volts. At $t = 0$ the charge on the capacitor and current in the circuit is zero. Find the charge and current at any time $t > 0$.

Solution: Let Q and I be the instantaneous charge and current respectively at time t.



By Kirchhoff's Laws, we have $2\frac{dI}{dt}+16I+\frac{Q}{0.02}=300$ ...... (1)

Since $I=\frac{dQ}{dt}$, so (1) becomes $2\frac{d^2Q}{dt^2}+16\frac{dQ}{dt}+50Q=300$

$$or,\quad \frac{d^2Q}{dt^2}+8\frac{dQ}{dt}+25Q=150 \quad \text{......} \quad (2)$$

with the initial conditions $Q(0)=0, \quad I(0)=Q'(0)=0$

Taking Laplace transformation in (2), we find

$$\mathcal{L}\{\frac{d^2Q}{dt^2}+8\frac{dQ}{dt}+25Q\}=\mathcal{L}\{150\}$$

$$\Rightarrow \left\{s^2q-sQ(0)-Q'(0)\right\}+8\left\{sq-Q(0)\right\}+25q=\frac{150}{s}$$

$$\Rightarrow s^2 q + 8sq + 25q = \frac{150}{s}$$

$$\Rightarrow \left(s^2 + 8s + 25\right)q = \frac{150}{s}$$

$$\Rightarrow q = \frac{150}{s\left(s^2 + 8s + 25\right)}$$

$$\Rightarrow q = \frac{6}{s} - \frac{6s + 48}{s^2 + 8s + 25} = \frac{6}{s} - \frac{6(s+4) + 24}{(s+4)^2 + 9}$$

$$\therefore q = \frac{6}{s} - \frac{6(s+4)}{(s+4)^2 + 3^2} - \frac{24}{(s+4)^2 + 3^2} \qquad \ldots\ldots \quad (3)$$

Taking inverse Laplace transformation in (3), we get

$$\mathcal{L}^{-1}\{q\} = \mathcal{L}^{-1}\left\{ \frac{6}{s} - \frac{6(s+4)}{(s+4)^2 + 3^2} - \frac{24}{(s+4)^2 + 3^2} \right\}$$

$$\therefore Q = 6 - 6e^{-4t}\cos 3t - 8e^{-4t}\sin 3t$$

The current of the circuit is

$$I = \frac{dQ}{dt} = 24e^{-4t}\cos 3t + 32e^{-4t}\sin 3t + 18\,e^{-4t}\sin 3t - 24e^{-4t}\cos 3t = 50\,e^{-4t}\sin 3t$$

Solving Electrical Circuits Problem

Problem: From the theory of electrical circuits we know, $i = C\frac{dv}{dt}$
where C is the capacitance, i = i(t) is the electric current , and v = v(t) is the voltage.
We have to find the correct expression for the complex impedance of a capacitor.

Solution:

- Taking the Laplace transform of this equation, we obtain, $I(s) = C(sV(s) - V_o)$,

  Where, $I(s) = \mathcal{L}\{i(t)\}$, $V(s) = \mathcal{L}\{v(t)\}$, and $V_o = v(t)|_{t=0}$.

- Solving for V(s) we have $V(s) = \frac{I(s)}{sC} + \frac{V_o}{s}$.

- We know, $Z(s) = \left.\frac{V(s)}{I(s)}\right|_{V_o=0}$.

  So we find:

$$Z(s) = \frac{1}{sC},$$

which is the correct expression for the complex impedance of a capacitor.

Question: An inductor of 3 henrys, a resistor of 30 ohms and an electromotive force
(e.m.f) of 150

volts. At $t = 0$ the current in the circuit is zero. Find the current at any time, $t>0$.

Question: An inductor of 2 henrys, a resistor of 16 ohms and a capacitor of 0.02 farads are connected in series with an electromotive force (e.m.f) of 100 volts. At $t = 0$ the charge on the capacitor and current in the circuit is zero. Find the charge and current at any time $t > 0$.

Week 7

Topics: Beam related problem

Pages (57)

APPLICATIONS TO BEAMS

18. A beam which is hinged at its ends $x = 0$ and $x = l$ [see Fig. 3-13] carries a uniform load W_0 per unit length. Find the deflection at any point.

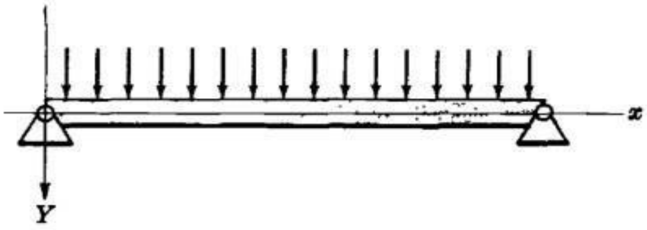


Fig. 3-13

The differential equation and boundary conditions are

$$\frac{d^4Y}{dx^4} = \frac{W_0}{EI} \qquad 0 < x < l \qquad (1)$$

$$Y(0) = 0, \; Y''(0) = 0, \; Y(l) = 0, \; Y''(l) = 0 \qquad (2)$$

Taking Laplace transforms of both sides of (1), we have, if $y = y(s) = \mathcal{L}\{Y(x)\}$,

$$s^4 y - s^3 Y(0) - s^2 Y'(0) - s Y''(0) - Y'''(0) = \frac{W_0}{EIs} \qquad (3)$$

Using the first two conditions in (2) and the unknown conditions $Y'(0) = c_1$, $Y'''(0) = c_2$, we find

$$y = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{W_0}{EIs^5}$$

Then inverting,

$$Y(x) = c_1 x + \frac{c_2 x^3}{3!} + \frac{W_0}{EI}\frac{x^4}{4!} = c_1 x + \frac{c_2 x^3}{6} + \frac{W_0 x^4}{24EI}$$

From the last two conditions in (2), we find

$$c_1 = \frac{W_0 l^3}{24EI}, \qquad c_2 = -\frac{W_0 l}{2EI}$$

Thus the required deflection is

$$Y(x) = \frac{W_0}{24EI}(l^3 x - 2lx^3 + x^4) = \frac{W_0}{24EI}x(l-x)(l^2 + lx - x^2)$$

Fourier Series

Week 8

Topics: Fourier analysis

Pages (58-71)

Fourier Series

Periodic Functions

The Mathematic Formulation

- Any function that satisfies

$$f(t) = f(t + T)$$

where T is a constant and is called the *period* of the function.

Example:

$$f(t) = \cos\frac{t}{3} + \cos\frac{t}{4}$$

Find its period.

$$f(t) = f(t+T) \Rightarrow \cos\frac{t}{3} + \cos\frac{t}{4} = \cos\frac{1}{3}(t+T) + \cos\frac{1}{4}(t+T)$$

Fact: $\cos\theta = \cos(\theta + 2m\pi)$

$$\frac{T}{3} = 2m\pi, \quad \frac{T}{4} = 2n\pi \Rightarrow T = 6m\pi, \quad T = 8n\pi \Rightarrow T = 24\pi$$

smallest T

Example:

$$f(t) = \cos\omega_1 t + \cos\omega_2 t$$

Find its period.

$$f(t) = f(t+T)$$

$\Rightarrow$

$$\cos\omega_1 t + \cos\omega_2 t = \cos\omega_1(t+T) + \cos\omega_2(t+T)$$

$$\omega_1 T = 2m\pi$$

$$\omega_2 T = 2n\pi$$

$\Rightarrow$ $\dfrac{\omega_1}{\omega_2} = \dfrac{m}{n}$ $\Rightarrow$ $\dfrac{\omega_1}{\omega_2}$ must be a rational number

Example:

$$f(t) = \cos 10t + \cos(10+\pi)t$$

Is this function a periodic one?

$$\frac{\omega_1}{\omega_2} = \frac{10}{10+\pi}$$

not a rational number

Some Important Functions:

Periodic Functions:

A function $f(x)$ is said to have a period P or to be periodic with periodic P if for all x, $f(x+P)=f(x)$, where P is a positive constant. The least value of $P>0$ is called the least period or simply the period of $f(x)$.

Ex1: The functions $\sin x$ and $\cos x$ has periods 2π, 4π, 6π, ….. However, 2π is the least period or periods of $\sin x$ and $\cos x$.

Ex2: The period of $\tan x$ is π.

Ex.3: A costant has any positive number as a period.

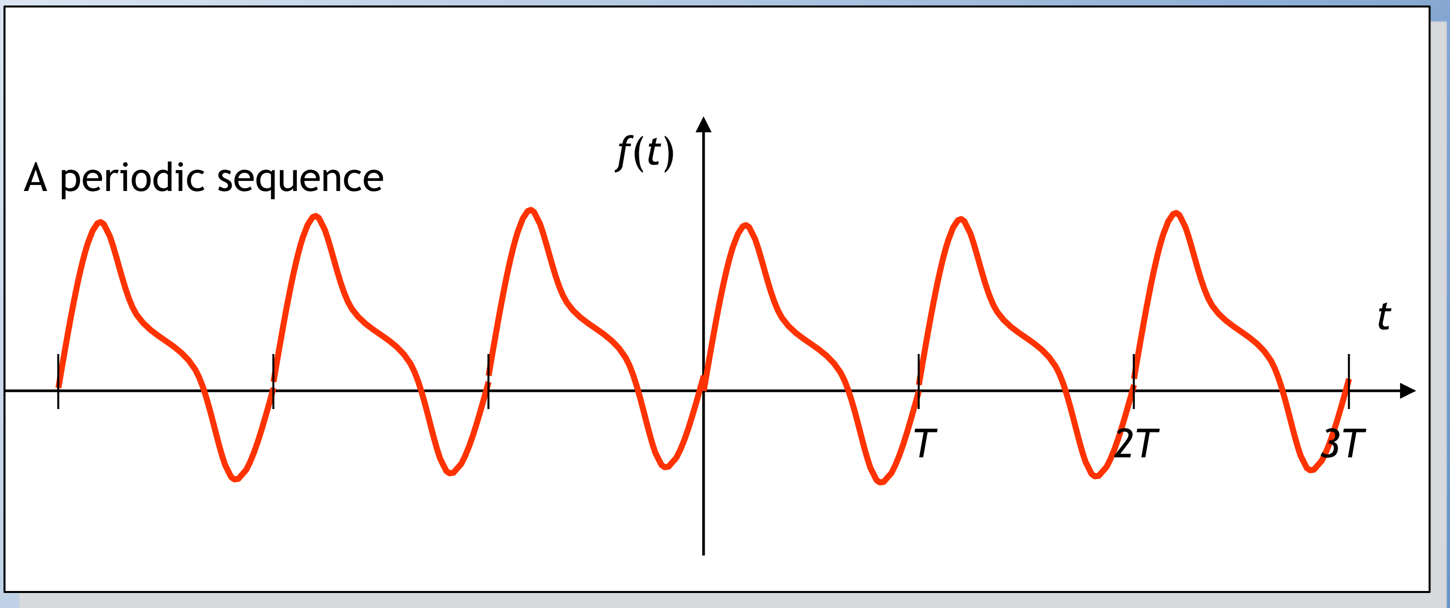
Piecewise Continuous Functions:

A function $f(x)$ is said to be piecewise continuous in an interval (i) the interval can be divided into a finite number of subintervals in each of which $f(x)$ is continuous and (ii) the limits of $f(x)$ as x approaches the endpoints of each subintervals are finite.

Ex: $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$ is a piecewise continuous function.

Introduction

- Decompose a periodic input signal into *primitive periodic components*.



Synthesis

$$f(t) = \underbrace{\frac{a_0}{2}}_{\text{DC Part}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos\frac{2\pi n t}{T}}_{\text{Even Part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{2\pi n t}{T}}_{\text{Odd Part}}$$

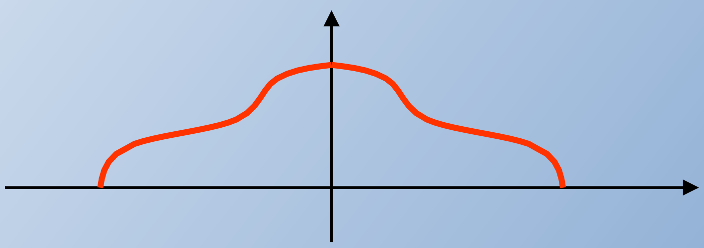
T is a period of all the above signals

Let $\omega_0 = 2\pi / T$.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Fourier Coefficients of Even Functions

$$f(t) = f(-t)$$

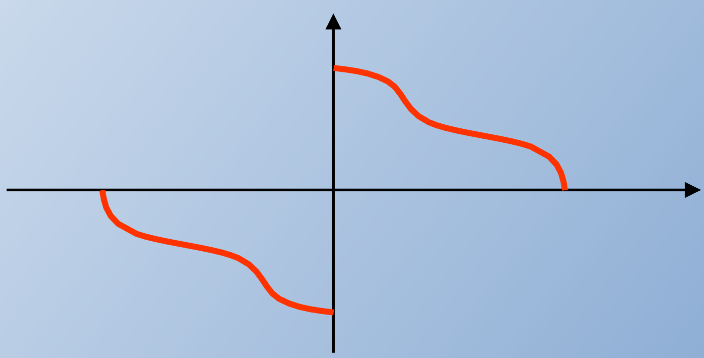


$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt$$

Fourier Coefficients of Odd Functions

$$f(t) = -f(-t)$$



$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)dt$$

Even and Odd Functions:

A function $f(x)$ is called even function if $f(-x)=f(x)$ and is called odd function if $f(-x)=-f(x)$.

Ex: x^2, x^4, x^6, $cosx$, $secx$ are even functions. Ex: x^3, x^5, x^7, $sinx$, $tan3x$ are odd functions.

Fourier Series:

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L)=f(x)$, i.e. assume that $f(x)$ has the period $2L$. The fourier series or fourier expansion corresponding to $f(x)$ is defined to be

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

Where the fourier coefficients a_0, a_n and b_n are

$$\begin{cases} a_0 = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x)\, dx \\ a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos \dfrac{n\pi x}{L}\, dx \qquad , n = 0,1,2\ldots\ldots \\ b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \sin \dfrac{n\pi x}{L}\, dx \end{cases}$$

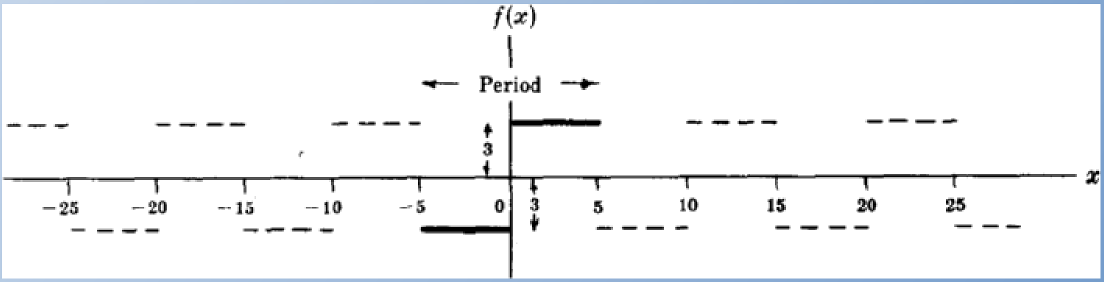
Problems: Graph each of the following functions:

$$(a)\ f(x)=\begin{cases}3 & 0<x<5\\ -3 & -5<x<0\end{cases}\quad Period=10$$

$$(b)\ f(x)=\begin{cases}\sin x & 0\le x\le \pi\\ 0 & \pi<x<2\pi\end{cases}\quad Period=2\pi$$

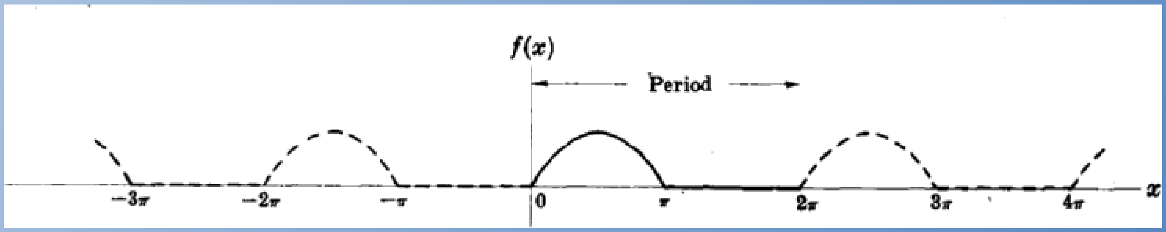
$$(c)\ f(x)=\begin{cases}0 & 0\le x<2\\ 1 & 2\le x<4\\ 0 & 4\le x<6\end{cases}\quad Period=6$$

Solution: (a)



Since the period is 10, the portion of the graph in *-5<x<5* which is indicated heavily is extended periodically outside of the range which is indicated in dashed. It is noted that *f(x)* is not defined at *x=0, 5, -5, 10, -10, 15, -15, 20*…These are the discontinuous point of *f(x)*.

(b)



Since the period is 2π, the portion of the graph in $0<x<2\pi$ which is indicated heavily is extended periodically outside of the range which is indicated in dashed. It is noted that *f(x)* is defined for all *x,* and is continuous everywhere.

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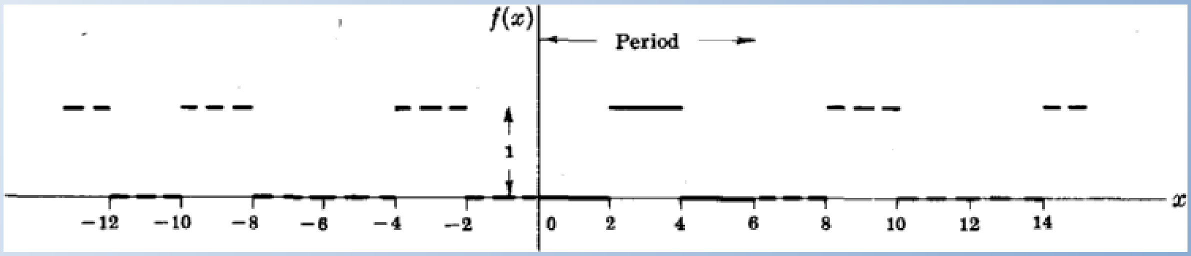


Since the period is 6, the portion of the graph in $0<x<6$ which is indicated heavily is extended periodically outside of the range which is indicated in dashed. It is noted that *f(x)* is defined for all *x*, and $x=-2, 2, -4, 4, -8, 8$ …are the discontinuous point of *f(x)*.s

Week 9
Topics: Graph of functions Pages (71-73)

Problems:

Classify each of the following functions according as they are even, odd or neither even nor odd.

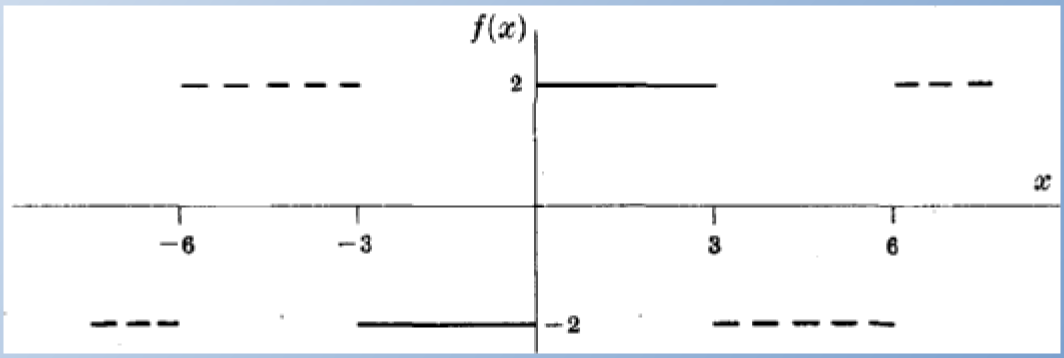
$$(a)\ f(x)=\begin{cases}2 & 0<x<3\\ -2 & -3<x<0\end{cases}\qquad Period=6$$

$$(b)\ f(x)=\begin{cases}\cos x & 0\le x\le \pi\\ 0 & \pi<x<2\pi\end{cases}\qquad Period=2\pi$$

$$(c)\ f(x)=x(10-x),\qquad 0<x<10\qquad Period=6$$

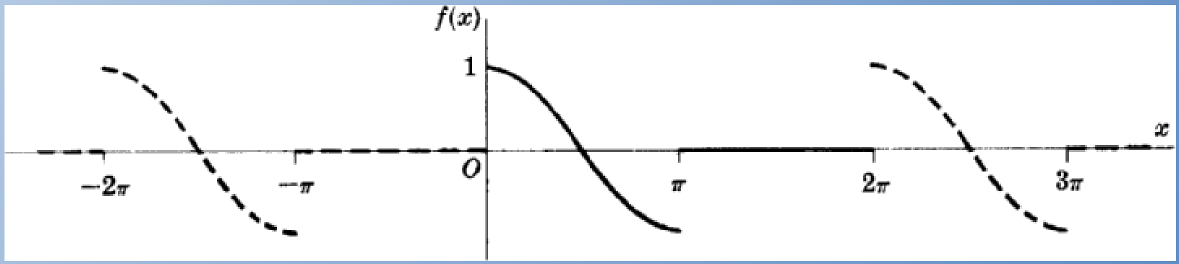
Solution:

(a) The graphical representation of the given function is



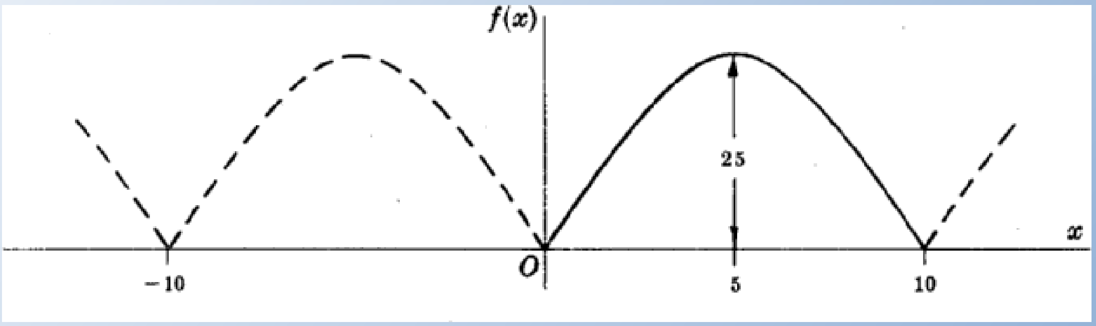
From the above figure we can see that the function is symmetric about the origin. So, it is seen from the figure that *f(-x)=-f(x)*, Hence the function is odd.

(b) The graphical representation of the given function is as follows



From the above figure we can see that the function is neither even nor odd.

© The graphical representation of the given function is



From the above figure we can see that the function is symmetric about y-axis. So, it is seen from the figure that *f(-x)=f(x)*, Hence the function is even.

Exercise1: Graph each of the following functions and classify them according as they are even, odd or neither even nor odd.

$(a)\ f(x)=\begin{cases}8 & 0<x<2\\ -8 & 2<x<4\end{cases}\quad Period=4$

$(b)\ f(x)=\begin{cases}-x & -4\le x\le 0\\ x & 0\le x\le 4\end{cases}\quad Period=8$

$(c)\ f(x)=4x,\quad 0<x<10\quad Period=10$

$(d)\ f(x)=\begin{cases}2x & 0\le x\le 3\\ 0 & -3\le x<0\end{cases}\quad Period=6$

Week 10

Topics: Dirichlet condition, Parsival's Identity Pages (73-75)

Dirichlet conditions for Fourier series

A set of Dirichlet conditions for the convergence of Fourier series are:

(1) a function "f" must be absolutely integrable over a period.

(2) a function "f" has bounded variation over one time period. The functions with bounded variations can have

(i) at most a countably infinite number of maxima and minima, and

(ii) at most a countably infinite number of finite discontinuities.

Dirichlet conditions for Fourier transform

A set of Dirichlet conditions for the convergence of Fourier transform are:

(1) a function "f" is absolutely integrable over the entire duration of time.

(2) a function "f" has bounded variation over the entire duration of time. The functions with bounded variations can contain (i) at most a countably infinite number of maxima and minima, and (ii) at most a countably infinite number of finite discontinuities.

Dirichlet conditions are sufficient but not necessary conditions.

Parseval's Identity: Let the Fourier series corresponding to *f(x)* converges

uniformly in *(-L,L)*, then the Parsival's Identity is

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Where a_0, a_n and b_n are Fourier coefficients respectively.

Parseval's Identity: Let the Fourier series corresponding to *f(x)* converges

uniformly in *(-L,L)*, then the Parsival's Identity is

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Where a_0, a_n and b_n are Fourier coefficients respectively.

Proof:

$$If\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \text{............}(1)$$

Then multiplying (1) by *f(x)* and integrating term by term from –L to L we get

$$\int_{-L}^{L}\{f(x)\}^2 = \frac{a_0}{2}\int_{-L}^{L} f(x) + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} + b_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\right)$$

$$= \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$So,\ \frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Where we have used the results

$$\int_{-L}^{L} f(x)dx = La_0,\ \int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = La_n,\ \int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = Lb_n \text{......}(2)$$

Is obtained from the Fourier Coefficients. Hence the Parsival’s identity is proved.

Week 11

Topics: mathematics of parsival's Identity

Pages (76-80)

Problem: (a) Expand *f(x)=x, 0<x<2* in a half range cosine series.(b) Write
Parsival's Identity corresponding to the Fourier series of (a).(c) Determine fom (b)

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots+\frac{1}{n^4}+\ldots$$

the sum S of the series

Solution:

(a) Extend the definition of the given function to that of the even function of period 4 which is shown in the below figure. This is sometimes called the even extension of *f(x)*. Then 2*L*=4, *L*=2.

Thus $b_n=0$,

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx=\frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$=\left\{(x)\left(\frac{2\sin n\pi x}{n\pi\quad 2}\right)-(1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2=\frac{-4}{n^2\pi^2}(\cos n\pi-1)\text{ if } n\neq 0$$

$$If\ \ n=0,\ a_0=\int_0^2 x\,dx=2$$

$$Then\ f(x)=1+\sum_{n=1}^{\infty}\frac{-4}{n^2\pi^2}(\cos n\pi-1)\cos\frac{n\pi x}{2}$$

$$=1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+\ldots\ldots\right)$$

(b) From (a) we get,

$$L=2, a_0=2, a_n=\frac{4}{n^2\pi^2}(\cos n\pi-1), n\neq 0; b_n=0$$

Then Parseval's identity
becomes

$$\frac{1}{2}\int_{-2}^{2}\{f(x)\}^2 = \frac{1}{2}\int_{-2}^{2} x^2 dx = \frac{2^2}{2} + \sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}(\cos n\pi - 1)$$

$$or \quad \frac{8}{3} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right)$$

$$i.e \quad \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}$$

© Here,

$$s = \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\right)$$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots\ldots\right)$$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\right)$$

$$= \frac{\pi^4}{96} + \frac{S}{16}$$

$i.e$ from which, $S = \frac{\pi_4}{90}$

Exercise5: (a) Expand *f(x)=x, 0<x<2* in a half range sine series.(b) Write Parsival's Identity corresponding to the Fourier series of (a).

Exercise6: (a) Expand *f(x)=x, 0<x<4* in a half range cosine series.(b) Write Parsival's Identity corresponding to the Fourier series of (a).

Exercise7: (a) Expand $f(x)=x,\ 0<x<4$ in a half range sine series.(b) Write

Parsival's Identity corresponding to the Fourier series of (a).

Week 12

Topics: Fourier Series Related mathematics

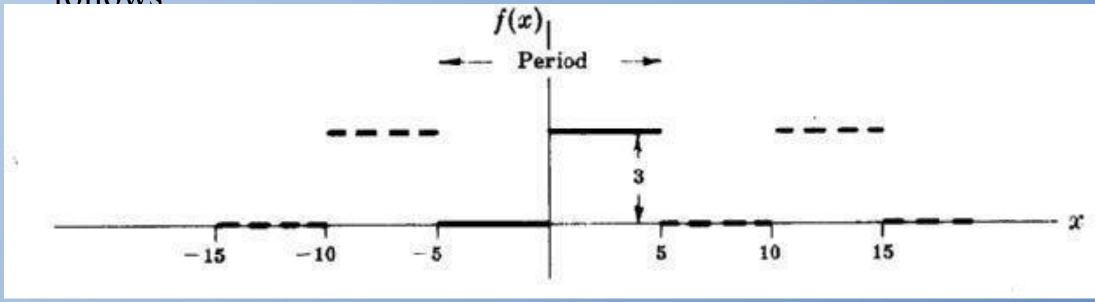
Pages (78-80)

Problem:

(a) Find the Fourier coefficients corresponding to the function

$$f(x)=\begin{cases}0 & -5<x<0\\ 3 & 0<x<5\end{cases} \qquad Period=10$$

(b) Write the corresponding Fourier series. Solution: The graph of $f(x)$ is as follows



(a) Period$=2L=10$ and $L=5$. Choose the interval c to $c+2L$ as -5 to 5, so that $c=-5$.
Then

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0} 0.\cos\frac{n\pi x}{5}dx + \int_0^5 3.\cos\frac{n\pi x}{5}dx\right\}$$

$$= \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}dx = \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \quad if\ n \neq 0$$

$$If\ n = 0,\ a_n = a_0 = \frac{1}{5}\int_0^5 3.\cos\frac{n\pi.0}{5}dx = \frac{3}{5}\int_0^5 3.dx = 3$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0} 0.\sin\frac{n\pi x}{5}dx + \int_0^5 3.\sin\frac{n\pi x}{5}dx\right\}$$

$$= \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}dx = \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3(1-\cos n\pi)}{n\pi}$$

(b) The corresponding Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{L} + b_n Sin\frac{n\pi x}{L}\right)$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty}\left(\frac{3}{n\pi}(1-Cos n\pi)Sin\frac{n\pi x}{5}\right)$$

$$= \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \ldots\right)$$

Exercise2: Graph each of the following functions, find the Fourier coefficients

$(\text{i})\ f(x)=\begin{cases}0 & -5<x<0\\ 3 & 0<x<5\end{cases}\quad Period=10 \qquad (\text{ii})\ f(x)=\begin{cases}8 & -2<x<0\\ 0 & 0<x<2\end{cases}\quad Period=4$

$(\text{iii})\ f(x)=\begin{cases}0 & -3<x<0\\ 4 & 0<x<3\end{cases}\quad Period=6 \qquad (\text{iv})\ f(x)=\begin{cases}-2 & -3<x<0\\ 2 & 0<x<3\end{cases}\quad Period=6$

$(\text{v})\ f(x)=\begin{cases}-3 & -5<x<0\\ 3 & 0<x<5\end{cases}\quad Period=10 \qquad (\text{vi})\ f(x)=\begin{cases}8 & -4<x<0\\ -8 & 0<x<4\end{cases}\quad Period=8$

Week 13 Topics: Half range Fourier Series
Pages (80-89)

Half range Fourier sine or cosine Series:

A half-range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half-range series corresponding to a given function is desired, the function is generally defined in the interval *(0, L)* and then the function is specified as odd or even. In such case, we have

$$\begin{cases} a_n=0, \quad b_n=\dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\dfrac{n\pi x}{L}\,dx \quad \text{for half-range sine series} \\ b_n=0, \quad a_n=\dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\dfrac{n\pi x}{L}\,dx \quad \text{for half-range cosine series} \end{cases}$$

<u>Problem3:</u> Expand *f(x)=x, 0<x<2* , in a half range (a) sine series, (b) cosine series. Solution:

(a) Extend the definition of the given function to that of the odd function of period 4 which is shown in the below figure. This is sometimes called the odd extension of *f(x)*. Then $2L=4$, $L=2$.

Thus $a_n=0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

$$\text{Then } f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \ldots\ldots\right)$$

(b) Extend the definition of the given function to that of the even function of period 4 which is shown in the below figure. This is sometimes called the even extension of $f(x)$. Then $2L=4$, $L=2$.

Thus $b_n=0$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n^2\pi^2}(\cos n\pi - 1) \text{ if } n \neq 0$$

$$\text{If } n = 0,\ a_0 = \int_0^2 x\,dx = 2$$

$$\text{Then } f(x) = 1 + \sum_{n=1}^{\infty}\frac{-4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$$

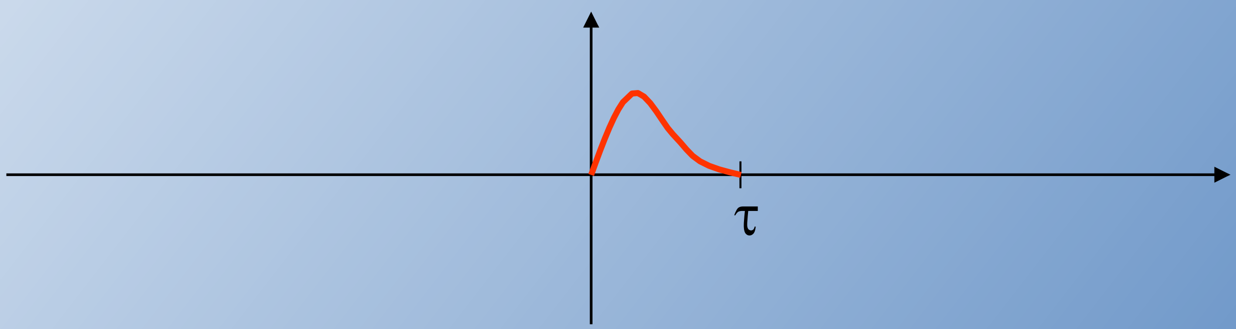
$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right)$$

It should be noted that although both series of (a) and (b) represent *f(x)* in the interval *0<x<2*, the second series converge more rapidly.

<u>Exercise 4:</u> Expand the followings functions in a half range (a) sine series, (b) cosine series.

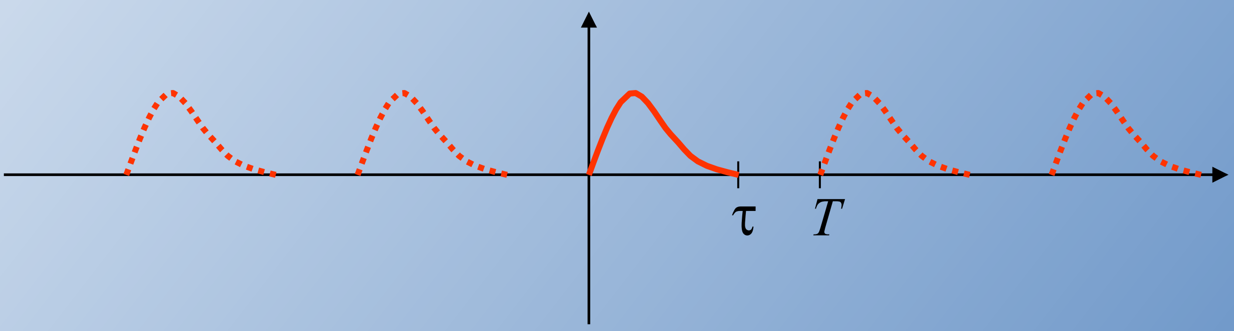
$$(i)\, f(x) = 4x, \qquad 0 < x < 4$$
$$(i)\, f(x) = ax, \qquad 0 < x < 2 \quad \textit{where a is any arbitrary constant.}$$
$$(i)\, f(x) = x, \qquad 0 < x < \pi$$

Non-Periodic Function Representation



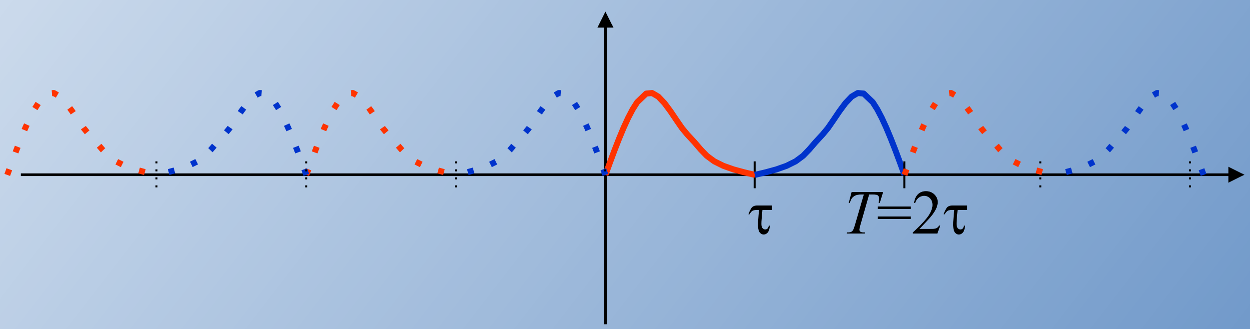
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Without Considering Symmetry



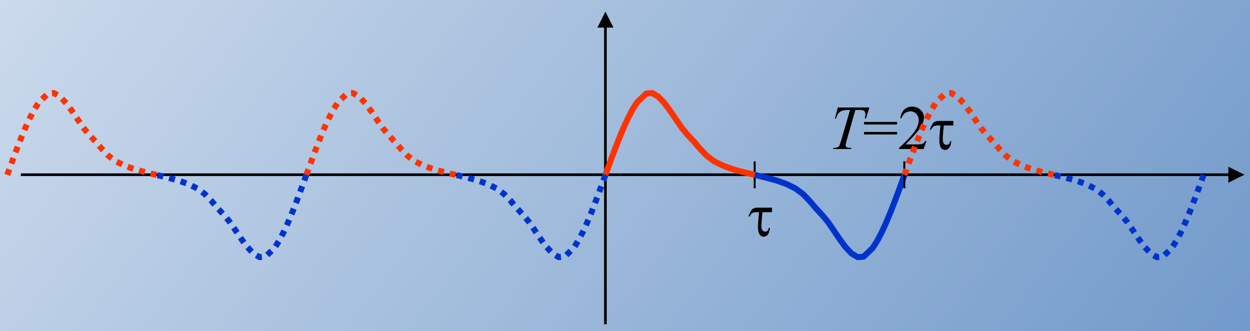
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Even Symmetry



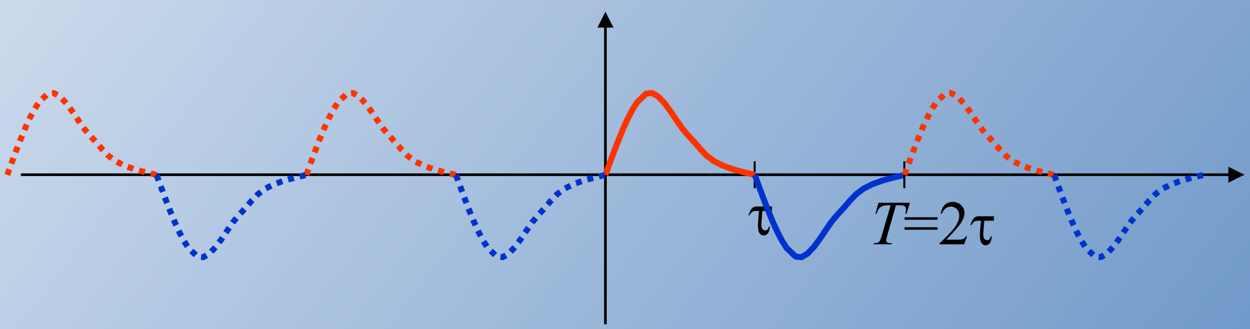
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Odd Symmetry



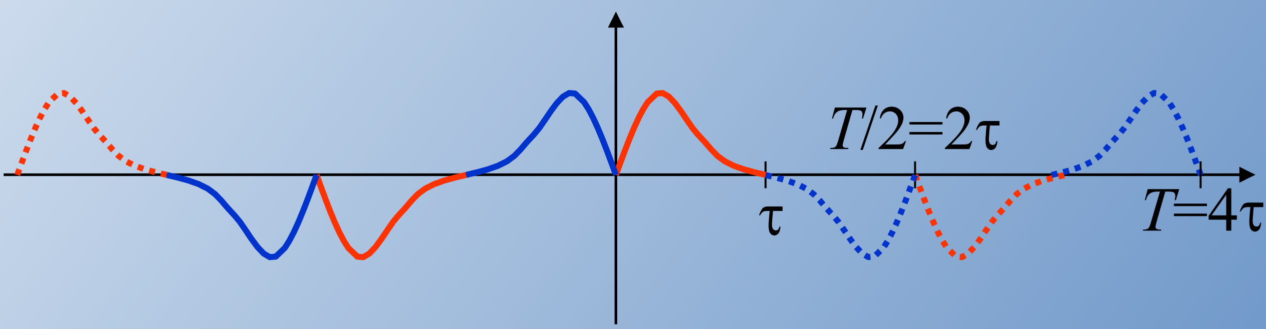
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Half-Wave Symmetry



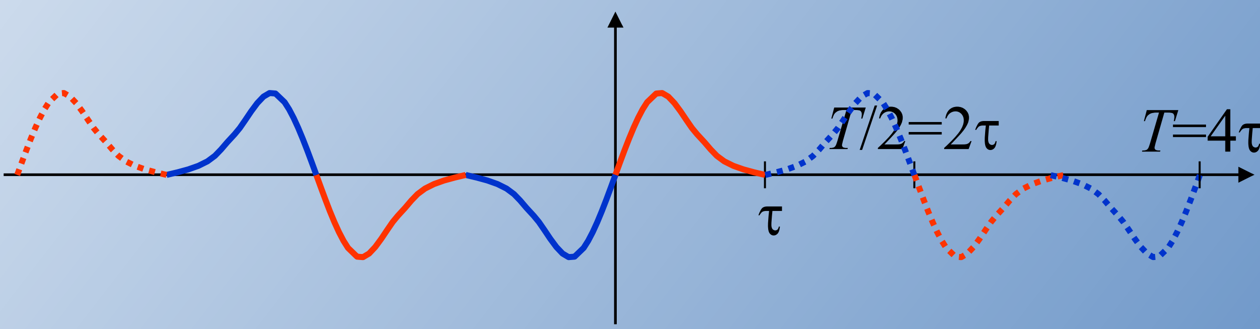
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Even Quarter-Wave Symmetry



- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into
Odd Quarter-Wave Symmetry



- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Week 14

Topics: Orthogonality

Pages (90-100)

Orthogonality:

Orthogonality is a fundamental concept in Fourier series, which are used to break down periodic functions into simpler terms:

- **Definition**: Two functions are orthogonal on an interval if their inner product is zero. The inner product is defined as the integral of the product of the two functions over the interval:

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0$$

Orthogonal sets

A set of functions is orthogonal if any two functions in the set are orthogonal. For example, the set of functions $(1, cosx, cos2x, cos3x)$ is orthogonal in the interval (-π π).

Orthogonal Functions

- Call a set of functions $\{\phi_k\}$ *orthogonal* on an interval $a < t < b$ if it satisfies

$$\int_a^b \phi_m(t)\phi_n(t)dt = \begin{cases} 0 & m \neq n \\ r_n & m = n \end{cases}$$

Orthogonal set of Sinusoidal Functions

Define $\omega_0 = 2\pi / T$.

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

We now prove this one

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0, \quad \text{for all } m \text{ and } n$$

Proof

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt \qquad m \neq n$$

$$= \frac{1}{2}\int_{-T/2}^{T/2}\cos[(m+n)\omega_0 t]dt + \frac{1}{2}\int_{-T/2}^{T/2}\cos[(m-n)\omega_0 t]dt$$

$$= \frac{1}{2}\frac{1}{(m+n)\omega_0}\sin[(m+n)\omega_0 t]\Big|_{-T/2}^{T/2} + \frac{1}{2}\frac{1}{(m-n)\omega_0}\sin[(m-n)\omega_0 t]\Big|_{-T/2}^{T/2}$$

$$= \frac{1}{2}\frac{1}{(m+n)\omega_0}\underbrace{2\sin[(m+n)\pi]}_{0} + \frac{1}{2}\frac{1}{(m+n)\omega_0}\underbrace{2\sin[(m-n)\pi]}_{0}$$

$$= 0$$

Proof

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\cos^2\alpha = \frac{1}{2}[1+\cos 2\alpha]$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt$$

m* = *n

$$= \int_{-T/2}^{T/2}\cos^2(m\omega_0 t)dt = \frac{1}{2}\int_{-T/2}^{T/2}[1+\cos 2m\omega_0 t]dt$$

$$= \frac{1}{2}t\Big|_{-T/2}^{T/2} + \underbrace{\frac{1}{4m\omega_0}\sin 2m\omega_0 t]\Big|_{-T/2}^{T/2}}_{0}$$

$$= \frac{T}{2}$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

Orthogonal set of Sinusoidal Functions

Define $\omega_0 = 2\pi / T$.

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0, \quad \text{for all } m \text{ and } n$$

$$\left\{ \begin{array}{l} 1, \\ \cos\omega_0 t,\ \cos 2\omega_0 t,\ \cos 3\omega_0 t, \cdots \\ \sin\omega_0 t,\ \sin 2\omega_0 t,\ \sin 3\omega_0 t, \cdots \end{array} \right\}$$

an orthogonal set.

Decomposition

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt \qquad n = 1,2,\ldots$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt \qquad n = 1,2,\ldots$$

Proof

Use the following facts:

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)dt = 0, \qquad m \neq 0$$

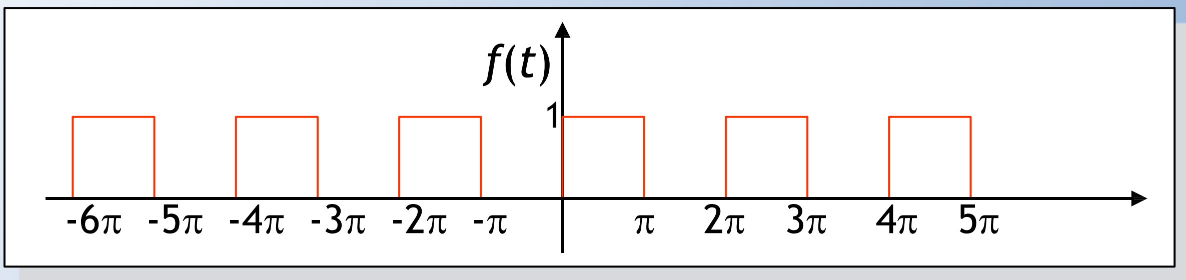
$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = 0, \quad \text{for all } m \text{ and } n$$

Example (Square Wave)



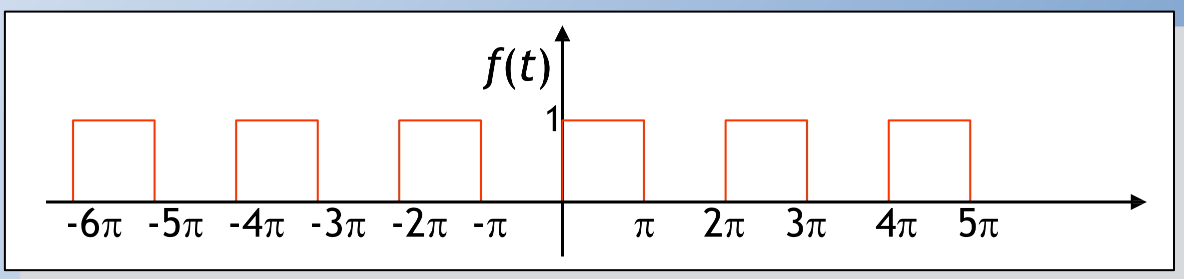
$$a_0 = \frac{2}{2\pi}\int_0^{\pi} 1\,dt = 1$$

$$a_n = \frac{2}{2\pi}\int_0^{\pi} \cos nt\,dt = \frac{1}{n\pi}\sin nt\Big|_0^{\pi} = 0 \qquad n = 1, 2, \ldots$$

$$b_n = \frac{2}{2\pi}\int_0^{\pi} \sin nt\,dt = -\frac{1}{n\pi}\cos nt\Big|_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - 1) = \begin{cases} 2/ & n = 1, 3, 5, \ldots \\ 0 & n = 2, 4, 6, \ldots \end{cases}$$

Example (Square Wave)

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$$



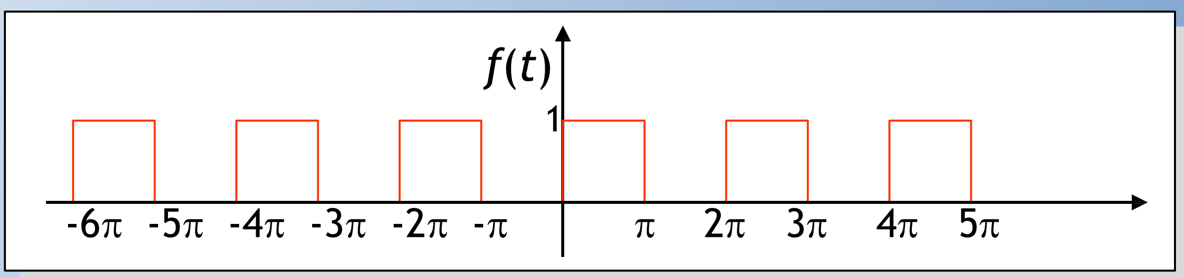
$$a_0 = \frac{2}{2\pi}\int_0^\pi 1\,dt = 1$$

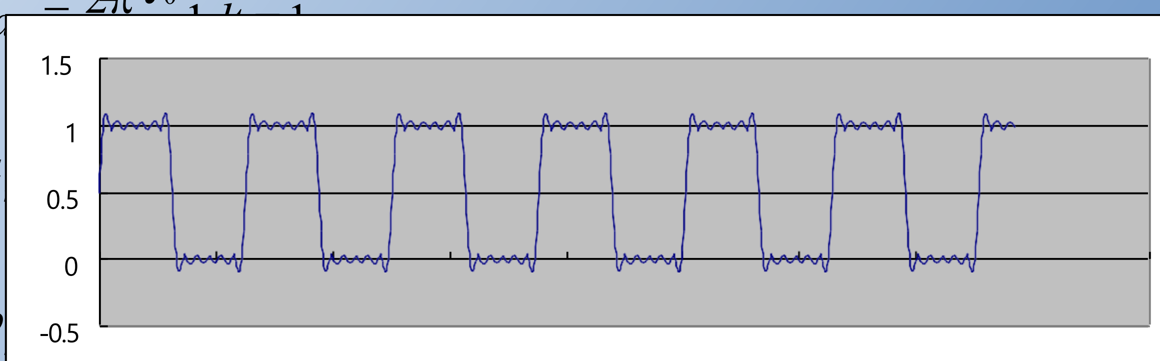
$$a_n = \frac{2}{2\pi}\int_0^\pi \cos nt\,dt = \frac{1}{n\pi}\sin nt\Big|_0^\pi = 0 \qquad n = 1,2,\dots$$

$$b_n = \frac{1}{2\pi}\int_0^\pi \sin nt\,dt = -\frac{1}{n\pi}\cos nt\Big|_0^\pi = -\frac{1}{n\pi}(\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1,3,5,\dots \\ 0 & n = 2,4,6,\dots \end{cases}$$

Example (Square Wave)

$$f(t)=\frac{1}{2}+\frac{2}{\pi}\left(\sin t+\frac{1}{3}\sin 3t+\frac{1}{5}\sin 5t+\cdots\right)$$





$n = 1,3,5,\ldots$

$n = 2,4,6,\ldots$

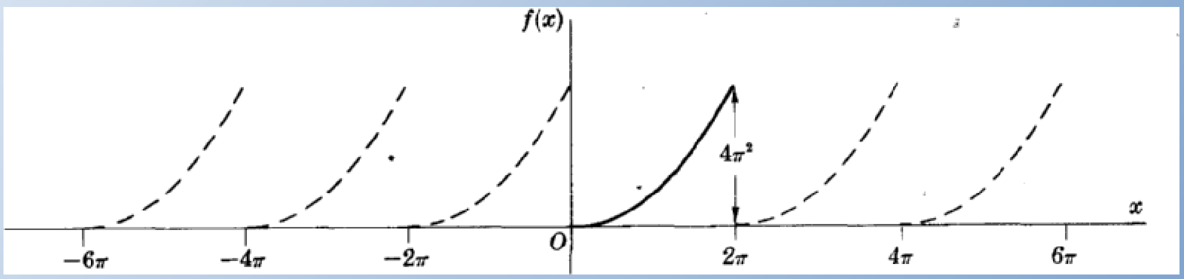
Week 15

Topics: Fourier Integrations

Pages (101-102)

Problem2: Expand $f(x)=x^2,\ 0<x<2\pi$, in a Fourier series if the period is 2π.

Solution: The graph of $f(x)$ with period 2π is as follows



Period=$2L=2\pi$ and $L=\pi$. Choosing c=0, we have

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos n\pi\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{4}{n^2},\ n \neq 0$$

$$\text{If } n = 0,\ a_n = a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{8\pi^2}{3}$$

$$b_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_{0}^{2\pi} x^2 \sin n\pi\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{-4\pi}{n}$$

$$Then\ f(x) = x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right) for\ 0 < x < 2\pi$$

<u>Exercise3:</u> Graph each of the following functions, and also find its corresponding Fourier series.

(i) $f(x) = 2x^2,\ 0 < x < 2\pi$

(ii) $f(x) = ax^2,\ 0 < x < 2\pi$, *where a is any arbitrary* constant.

(iii) $f(x) = x^2,\ 0 < x < \pi$

Week16

Topics: Applications

Pages (103-104)

Application:

Problem: Solve the boundary value problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2},\ u(0,t) = 10, u(3,t) = 40,\ u(x,0) = 25,\ |u(x,t)| < M$$

Solution: To solve the present problem assume that *u(x,t)=v(x,t)+ϕ(x,t)* where

ϕ(x,t) is to be suitably determined. In terms of *v(x,t)* the boundary value problem

becomes

$$\frac{\partial v}{\partial t} = 2\frac{\partial^2 v}{\partial x^2} + 2\phi''(x),\ v(0,t) + \phi(0) = 10, v(3,t) + \phi(3) = 40,\ v(x,0) + \phi(x) = 25,\ |v(x,t)| < M$$

This can be simplified by choosing

$\phi''(x) = 0,\ \phi(0) = 3,\ \phi(3) = 40$

From which e can find *ϕ(x)=10x+10*, o that the resulting boundary value

problem is

$$\frac{\partial v}{\partial t} = 2\frac{\partial^2 v}{\partial x^2},\ v(0,t) = 10, v(3,t) = 40,\ v(x,0) = 15-10x$$

We can find the solution of this problem is in the form

$$v(x,t) = \sum_{m=1}^{\infty} B_m e^{-2m^2\pi^2 t/9} \sin\frac{m\pi x}{3}$$

The last condition
yields

$$15 - 10x = \sum_{m=1}^{\infty} B_m \sin\frac{m\pi x}{3}$$

From which

$$B_m = \frac{2}{3}\int_0^3 (15-10x)\sin\frac{m\pi x}{3}dx = \frac{30}{m\pi}(\cos m\pi - 1)$$

Since $u(x,t) = v(x,t) + \phi(x,t)$, we have finally

$$u(x,t)=10x+10+\sum_{m=1}^{\infty}\frac{30}{m\pi}(\cos m\pi - 1)e^{-2m^2\pi^2 t/9}\sin\frac{m\pi x}{3}$$

as the required solution.

The term *10x+10* is the *steady-state* temperature, i.e. the temperature after a long
time has elapsed.

Week17

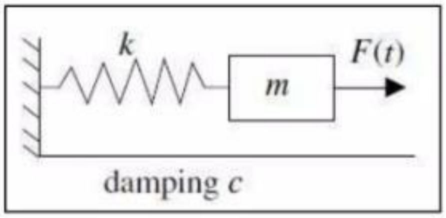
Topics: Applications

Pages (105-107)

Consider a mass-spring system as before, where we have a mass m on a spring with spring constant k, with damping c, and a force $F(t)$ applied to the mass.

Suppose the forcing function $F(t)$ is $2L$-periodic for some $L > 0$.

The equation that governs this particular setup is



$$mx''(t) + cx'(t) + kx(t) = F(t)$$

The general solution consists of the complementary solution x_c, which solves the associated homogeneous equation $mx'' + cx' + kx = 0$, and a particular solution of (1) we call x_p.

For $c > 0$,
the complementary solution x_c will decay as time goes by. Therefore, we are mostly interested in a particular solution x_p that does not decay and is periodic with the same period as F(t). We call this particular solution the steady periodic solution and we write it as x_{sp} as before. What will be new in this section is that we consider an arbitrary forcing function F(t) instead of a simple cosine.
For simplicity, let us suppose that $c = 0$. The problem with $c > 0$ is very similar. The equation

$$mx'' + kx = 0$$

has the general solution,

$$x(t) = A\cos(\omega t) + B\sin(\omega t);$$

Where,

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

Any solution to mx"(t) + kx(t) = F(t) is of the form

A cos(ωt) + B sin(ωt) + x_{sp}.

The steady periodic solution x_{sp} has the same period as F(t).

In the spirit of the last section and the idea of undetermined coecients we first write,

$$F(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L} t\right) + d_n \sin\left(\frac{n\pi}{L} t\right).$$

Then we write a proposed steady periodic solution x as,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right),$$

where a_n and b_n are unknowns. We plug x into the deferential equation and solve for a_n and b_n in terms of c_n and d_n.